Lengths of vectors, areas of parallelograms, and volumes of parallelepipeds

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The length of a vector **a**, the area of a parallelogram defined by two edge vectors **a** and **b**, and the volume of a parallelepiped[1] defined by three edge vectors **a**, **b**, and **c** are given by[2]

$$\begin{aligned} a \equiv \text{Length}(\mathbf{a}) &= \sqrt{|\mathbf{a}\cdot\mathbf{a}|} = \sqrt{\mathbf{a}\cdot\mathbf{a}}\,, \\ \text{Area}(\mathbf{a},\mathbf{b}) &= \sqrt{\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} \end{vmatrix}} = \sqrt{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{a})}\,, \\ \text{Volume}(\mathbf{a},\mathbf{b},\mathbf{c}) &= \sqrt{\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}} = \sqrt{\begin{matrix} (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{c}) + (\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{a}) + (\mathbf{a}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{a}) \\ -(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{b}) - (\mathbf{b}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{a}) - (\mathbf{c}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{a}) \end{matrix}}\,, \end{aligned} \tag{1}$$

where[3]

$$\begin{aligned} \mathbf{b}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b} &= (\text{length of } \mathbf{a})(\text{perpendicular projection of } \mathbf{b} \text{ along the direction of } \mathbf{a}) \\ &= (\text{length of } \mathbf{b})(\text{perpendicular projection of } \mathbf{a} \text{ along the direction of } \mathbf{b}) \\ &= (\text{length of } \mathbf{a})(\text{length of } \mathbf{b})(\text{cosine of the angle } \theta \text{ between the direction of } \mathbf{a} \text{ and the direction of } \mathbf{b}) \\ &= ab\cos\theta. \end{aligned} \tag{2}$$

For example, as shown in Figure 1, given a parallelepiped with edge vectors **a**, **b**, and **c** that meet at a corner and given that

$$\begin{aligned} &\text{length of } \mathbf{a} = 5 \text{ meters}, && \text{perpendicular projection of } \mathbf{b} \text{ along the direction of } \mathbf{a} = 2 \text{ meters}, \\ &\text{length of } \mathbf{b} = 3 \text{ meters}, && \text{perpendicular projection of } \mathbf{a} \text{ along the direction of } \mathbf{c} = -4 \text{ meters}, \\ &\text{length of } \mathbf{c} = 2 \text{ meters}, && \text{perpendicular projection of } \mathbf{b} \text{ along the direction of } \mathbf{c} = -1 \text{ meter}, \end{aligned} \tag{3}$$

the parallelepiped volume is

$$\text{Volume}(\mathbf{a},\mathbf{b},\mathbf{c}) = \sqrt{\begin{vmatrix} 25\text{ m}^2 & 10\text{ m}^2 & -8\text{ m}^2 \\ 10\text{ m}^2 & 9\text{ m}^2 & -2\text{ m}^2 \\ -8\text{ m}^2 & -2\text{ m}^2 & 4\text{ m}^2 \end{vmatrix}} = \sqrt{\begin{matrix} 900\text{ m}^6 + 160\text{ m}^6 + 160\text{ m}^6 \\ -100\text{ m}^6 - 576\text{ m}^6 - 400\text{ m}^6 \end{matrix}} = \sqrt{144\text{ m}^6} = 12\text{ m}^3. \tag{4}$$

If edge **c** were, instead, a fluid velocity of speed 2 m/s, or an electric field of strength 2 kg m/(C s^2), or a magnetic field of strength 2 kg/(C s), then the parallelepiped volume would be the fluid volume per time 12 m^3/s, or the electric flux 12 kg m^3/(C s^2), or the magnetic flux 12 kg m^2/(C s), respectively, through a parallelogram defined by edges **a** and **b**.

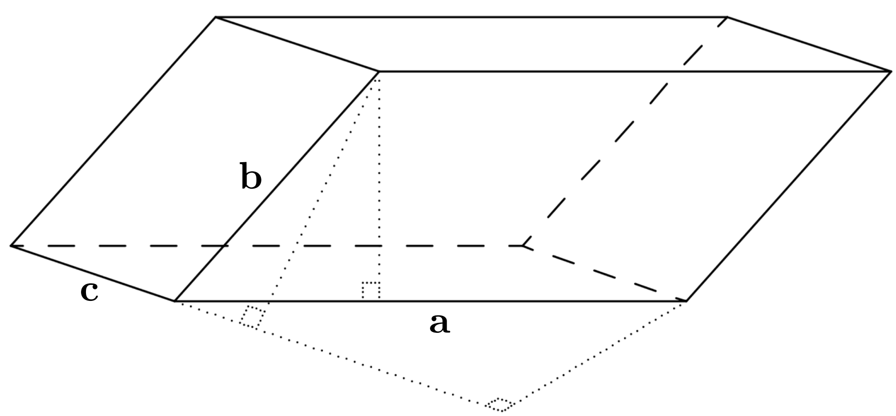

Figure 1 Parallelepiped with edge vectors **a**, **b**, and **c** as in Eq. (3) and with volume given by Eq. (4).

Equations (1) and (2) extend to higher dimensions and complex numbers, have many practical applications in science, engineering, and other areas of applied mathematics, and are important connections between geometry and algebra.

[1] A parallelepiped is a six-faced volume all six faces of which are parallelograms.

[2] The length, area, and volume are square roots of determinants of 1-by-1, 2-by-2, and 3-by-3 Gram matrices, whose matrix elements are scalar products (dot products) of edge vectors **a**, **b**, and **c**.

[3] This assumes that the scalar product $\mathbf{a}\cdot\mathbf{b}$ of the vector **a** with the vector **b** is a real number. If the scalar product is a complex number, then the scalar product $\mathbf{b}\cdot\mathbf{a}$ equals the complex conjugate of the scalar product $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{b}\cdot\mathbf{a} = \overline{\mathbf{a}\cdot\mathbf{b}}$.

Division of the edge vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ by their lengths gives vectors $\hat{\mathbf{a}} = \mathbf{a}/\text{Length}(\mathbf{a})$, $\hat{\mathbf{b}} = \mathbf{b}/\text{Length}(\mathbf{b})$, $\hat{\mathbf{c}} = \mathbf{c}/\text{Length}(\mathbf{c})$ of unit length, $\text{Length}(\hat{\mathbf{a}}) = 1$, $\text{Length}(\hat{\mathbf{b}}) = 1$, $\text{Length}(\hat{\mathbf{c}}) = 1$, called *direction* or *unit* vectors.

The area of a parallelogram defined by two direction vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is given by the magnitude of the sine of the angle θ between direction vector $\hat{\mathbf{a}}$ and direction vector $\hat{\mathbf{b}}$:

$$\text{Area}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \frac{\text{Area}(\mathbf{a}, \mathbf{b})}{\text{Length}(\mathbf{a})\text{Length}(\mathbf{b})} = \frac{1}{ab}\sqrt{\begin{vmatrix} a^2 & ab\cos\theta \\ ab\cos\theta & b^2 \end{vmatrix}} = \frac{\sqrt{a^2b^2 - a^2b^2\cos^2\theta}}{ab} = \sqrt{1-\cos^2\theta} = \sqrt{\sin^2\theta} = |\sin\theta| \leq 1. \quad (5)$$

The magnitude of the sine of the angle ϕ between edge vector $\mathbf{c}$ and the plane of the parallelogram defined by edge vector $\mathbf{a}$ and edge vector $\mathbf{b}$ is given by

$$|\sin\phi| = \frac{\text{Volume}(\mathbf{a}, \mathbf{b}, \mathbf{c})}{\text{Area}(\mathbf{a}, \mathbf{b})\text{Length}(\mathbf{c})} = \frac{\text{Volume}(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}})}{\text{Area}(\hat{\mathbf{a}}, \hat{\mathbf{b}})\text{Length}(\hat{\mathbf{c}})} \leq 1. \quad (6)$$

The volume of a parallelepiped defined by direction vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ is given by the product of Equations (5) and (6):

$$\text{Volume}(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}) = \frac{\text{Volume}(\mathbf{a}, \mathbf{b}, \mathbf{c})}{\text{Length}(\mathbf{a})\text{Length}(\mathbf{b})\text{Length}(\mathbf{c})} = |\sin\theta||\sin\phi| \leq 1. \quad (7)$$