Roots of a cubic equation[1]

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We wish to find the roots of the cubic equation

$$0 = ax^3 + bx^2 + cx + d = a(x - x_0)(x - x_1)(x - x_2) \tag{1}$$

for arbitrary real coefficients $a \neq 0$, b, c, d. The ratios b/a, c/a, d/a are given in terms of the roots, x_0, x_1, x_2, by

$$b/a = -(x_0+x_1+x_2)\,, \qquad c/a = x_0x_1+x_0x_2+x_1x_2, \qquad d/a = -x_0x_1x_2. \tag{2}$$

Division of Eq. (1) by a and the substitution

$$x = y - b/(3a) \tag{3}$$

give a cubic equation in y in which quadratic terms cancel,

$$0 = \left(y - \frac{b}{3a}\right)^3 + \frac{b}{a}\left(y - \frac{b}{3a}\right)^2 + \frac{c}{a}\left(y - \frac{b}{3a}\right) + \frac{d}{a} = y^3 - \left(\frac{b^2}{3a^2} - \frac{c}{a}\right)y + \left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\right) = y^3 - py + q, \tag{4}$$

where

$$p = \frac{b^2}{3a^2} - \frac{c}{a}, \qquad q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}. \tag{5}$$

For $p = 0$, Eq. (4) is solved by cube roots of $-q$ and the roots of Eq. (1) are given, according to Eqs. (3) and (5), by[2]

$$x_0 = (-q)^{\frac{1}{3}} - b/(3a)\,, \qquad x_1 = (-q)^{\frac{1}{3}}\left(-1+i\sqrt{3}\right)/2 - b/(3a)\,, \qquad x_2 = (-q)^{\frac{1}{3}}\left(-1-i\sqrt{3}\right)/2 - b/(3a)\,. \tag{6}$$

The root x_0 is real and, unless $q = 0$, the roots x_1 and x_2 are complex and conjugate to each other.

The cubic equation has three real roots if and only if the function $f(y) = y^3 - py + q$ has a local maximum $f_{\max}$ no less than zero and a local minimum $f_{\min}$ no greater than zero,

$$0 \le -f_{\max}f_{\min} = -f(y_{\max})\,f(y_{\min}) = -f\left(-\sqrt{p/3}\right)f\left(\sqrt{p/3}\right) = \left(\frac{2p}{3}\sqrt{\frac{p}{3}}+q\right)\left(\frac{2p}{3}\sqrt{\frac{p}{3}}-q\right) = 4p^3/27 - q^2. \tag{7}$$

The types of roots of the cubic equation are, in general, determined by the value of the *discriminant*

$$\Delta \equiv 4p^3 - 27q^2 \equiv b^2c^2/a^4 - 4b^3d/a^4 - 4c^3/a^3 + 18bcd/a^3 - 27d^2/a^2 \equiv (x_0-x_1)^2(x_0-x_2)^2(x_1-x_2)^2. \tag{8}$$

If $\Delta > 0$, then all three roots are real and none are equal; if $\Delta = 0$, then all three roots are real and two are equal; and if $\Delta < 0$, then one root is real and two roots are complex and conjugate to each other.

For $\Delta = 4p^3 - 27q^2 \ge 0$ and $p \neq 0$, p is positive and the substitution

$$y = 2\sqrt{p/3}\cos\theta \tag{9}$$

may be used with the identity

$$4\cos^3\theta = 3\cos\theta + \cos(3\theta) \tag{10}$$

to simplify Eq. (4) to

$$0 = y^3 - py + q = \frac{8p}{3}\sqrt{\frac{p}{3}}\cos^3\theta - 2p\sqrt{\frac{p}{3}}\cos\theta + q = \frac{2p}{3}\sqrt{\frac{p}{3}}\cos(3\theta) + q. \tag{11}$$

Equation (11) is solved by the angles

$$\theta_0 = \frac{1}{3}\arccos\left(-\frac{3q}{2p}\sqrt{\frac{3}{p}}\right), \qquad \theta_1 = \frac{1}{3}\arccos\left(-\frac{3q}{2p}\sqrt{\frac{3}{p}}\right) + \frac{2\pi}{3}, \qquad \theta_2 = \frac{1}{3}\arccos\left(-\frac{3q}{2p}\sqrt{\frac{3}{p}}\right) + \frac{4\pi}{3}, \tag{12}$$

and by all angles obtained by addition of integer multiples of 2π to θ_0, θ_1, and θ_2. The respective real roots of Eq. (1) are given, according to Eqs. (3), (5), (9), and (12), by

$$\begin{aligned}
x_0 &= 2\sqrt{\frac{p}{3}}\cos\theta_0 - \frac{b}{3a} = 2\sqrt{\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left(-\frac{3q}{2p}\sqrt{\frac{3}{p}}\right)\right] - \frac{b}{3a},\\
x_1 &= 2\sqrt{\frac{p}{3}}\cos\theta_1 - \frac{b}{3a} = 2\sqrt{\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left(-\frac{3q}{2p}\sqrt{\frac{3}{p}}\right) + \frac{2\pi}{3}\right] - \frac{b}{3a},\\
x_2 &= 2\sqrt{\frac{p}{3}}\cos\theta_1 - \frac{b}{3a} = 2\sqrt{\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left(-\frac{3q}{2p}\sqrt{\frac{3}{p}}\right) + \frac{4\pi}{3}\right] - \frac{b}{3a}.
\end{aligned} \tag{13}$$

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[1] Ivan S. Sokolnikof and Elizabeth S. Sokolnikoff, *Higher Mathematics for Engineers and Physicists*, McGraw-Hill, New York 1941, pages 86-91. The sign of the definition of p used by the Sokolnikoffs differs from that used here.

[2] The cube roots of unity are 1, $\left(-1+i\sqrt{3}\right)/2$, and $\left(-1-i\sqrt{3}\right)/2$ and $(-q)^{\frac{1}{3}}$ denotes the real cube root of $-q$. For $q > 0$, $(-q)^{\frac{1}{3}} = -q^{\frac{1}{3}} < 0$, and, for $q < 0$, $(-q)^{\frac{1}{3}} = -q^{\frac{1}{3}} > 0$, where $q^{\frac{1}{3}}$ denotes the real cube root of q.

For $\Delta = 4p^3 - 27q^2 < 0$ and $p \neq 0$, multiplication of Eq. (4) by w^3 and the substitution

$$y = w + p/(3w) \tag{14}$$

give a quadratic equation in w^3,

$$0 = w^3\left(y^3 - py + q\right) = w^3\left[\left(w + \frac{p}{3w}\right)^3 - p\left(w + \frac{p}{3w}\right) + q\right] = w^3\left(w^3 + q + \frac{p^3}{27w^3}\right) = w^6 + qw^3 + \frac{p^3}{27}, \tag{15}$$

which has two real roots,

$$w_+^3 = \frac{-q + \sqrt{q^2 - 4p^3/27}}{2}, \qquad w_-^3 = \frac{-q - \sqrt{q^2 - 4p^3/27}}{2}. \tag{16}$$

Values of w that solve Eq. (15) are cube roots of w_+^3 and w_-^3,

$$\begin{aligned}
w_{+0} &= \left(\frac{-q + \sqrt{q^2 - 4p^3/27}}{2}\right)^{\frac{1}{3}}, & w_{-0} &= \left(\frac{-q - \sqrt{q^2 - 4p^3/27}}{2}\right)^{\frac{1}{3}}, \\
w_{+1} &= \left(\frac{-q + \sqrt{q^2 - 4p^3/27}}{2}\right)^{\frac{1}{3}}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), & w_{-1} &= \left(\frac{-q - \sqrt{q^2 - 4p^3/27}}{2}\right)^{\frac{1}{3}}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right), \\
w_{+2} &= \left(\frac{-q + \sqrt{q^2 - 4p^3/27}}{2}\right)^{\frac{1}{3}}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right), & w_{-2} &= \left(\frac{-q - \sqrt{q^2 - 4p^3/27}}{2}\right)^{\frac{1}{3}}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right).
\end{aligned} \tag{17}$$

The respective roots of Eq. (1) are given, according to Eqs. (3), (5), (14), and (17), by

$$\begin{aligned}
x_0 &= w_{+0} + \frac{p}{3w_{+0}} - \frac{b}{3a} = w_{-0} + \frac{p}{3w_{-0}} - \frac{b}{3a}, \\
x_1 &= w_{+1} + \frac{p}{3w_{+1}} - \frac{b}{3a} = w_{-1} + \frac{p}{3w_{-1}} - \frac{b}{3a}, \\
x_2 &= w_{+2} + \frac{p}{3w_{+2}} - \frac{b}{3a} = w_{-2} + \frac{p}{3w_{-2}} - \frac{b}{3a}.
\end{aligned} \tag{18}$$

The root x_0 is real and the roots x_1 and x_2 are complex and conjugate to each other.