Lowest-weight and highest-weight hermirreps of $\mathrm{SO}(2,1)$, representation splitting, contraction to $\mathrm{HW}(1)$, and realization of $\mathrm{SO}(2,1)$ basis operators as functions of $\mathrm{HW}(1)$ basis operators
©Mark Loewe, Austin, 28 November 2020

The (complexified) Lie algebra of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ has a basis of three operators,

$$\begin{array}{ll} F_{01}, & F_{05}, \\ & F_{15}, \end{array} \tag{1}$$

that satisfy the commutation relations

$$[F_{ab},F_{cd}] \equiv F_{ab}F_{cd} - F_{cd}F_{ab} = -i\,(\eta_{ac}F_{bd} + \eta_{bd}F_{ac} - \eta_{ad}F_{bc} - \eta_{bc}F_{ad}), \qquad \text{for } a,b \in \{0,1,5\}, \tag{2}$$

where $F_{ab} = -F_{ba}$, $\eta_{00} = -\eta_{11} = \eta_{55} = 1$, and $\eta_{ab} = 0$ for $a \neq b$. A commutator of Eq. (2) is non-zero if and only if only one of the indices of F_{ab} equals one of the indices of F_{cd}. All nine commutators are

$$\begin{array}{lll} [F_{05},F_{05}] = 0, & [F_{01},F_{01}] = 0, & [F_{15},F_{15}] = 0, \\ [F_{01},F_{15}] = -iF_{05}, & [F_{05},F_{15}] = -iF_{01}, & [F_{05},F_{01}] = iF_{15}, \\ [F_{15},F_{01}] = iF_{05}, & [F_{15},F_{05}] = iF_{01}, & [F_{01},F_{05}] = -iF_{15}. \end{array} \tag{3}$$

A basis in the Lie algebra of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ that consists of operators which raise by 1, lower by 1, or leave unchanged the eigenvalue of F_{05} is given by the operators

$$F_+ = \tfrac{1}{\sqrt{2}}(F_{01} + iF_{15}), \quad F_{05}, \quad F_- = \tfrac{1}{\sqrt{2}}(F_{01} - iF_{15}). \tag{4}$$

The basis of Eq. (1) is given in terms of the basis of Eq. (4) by

$$\begin{array}{ll} F_{01} = \tfrac{1}{\sqrt{2}}(F_- + F_+), & F_{05}, \\ & F_{15} = \tfrac{i}{\sqrt{2}}(F_- - F_+). \end{array} \tag{5}$$

The commutation relations of the operators of Eq. (4) are

$$\begin{array}{lll} [F_{05},F_{05}] = 0, & [F_-,F_-] = 0, & [F_+,F_+] = 0, \\ [F_-,F_+] = F_{05}, & [F_{05},F_-] = -F_-, & [F_{05},F_+] = F_+, \\ [F_+,F_-] = -F_{05}, & [F_-,F_{05}] = F_-, & [F_+,F_{05}] = -F_+. \end{array} \tag{6}$$

The second-order operator

$$F_{05}F_{05} - F_{01}F_{01} - F_{15}F_{15} = F_{05}F_{05} - F_+F_- - F_-F_+ = F_{05}F_{05} - F_{05} - 2F_+F_- = F_{05}F_{05} + F_{05} - 2F_-F_+ \tag{7}$$

commutes with all operators

$$\mathcal{E} = cI + \sum_{a,b} c^{ab}F_{ab} + \sum_{a,b,c,d} c^{abcd}F_{ab}F_{cd} + \sum_{a,b,c,d,e,f} c^{abcdef}F_{ab}F_{cd}F_{ef} + \cdots, \qquad c, c^{ab}, c^{abcd}, c^{abcdef}, \ldots \in \mathcal{C}, \tag{8}$$

in the enveloping algebra $\mathcal{E}(\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}})$ of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$.

Let $|\mu\rangle$ be an eigenvector of the operator F_{05} with eigenvalue μ,

$$F_{05}|\mu\rangle = \mu|\mu\rangle\,. \tag{9}$$

If $F_+|\mu\rangle$ is not the zero vector, then it is an eigenvector of F_{05} with eigenvalue $\mu+1$,

$$F_{05}F_+|\mu\rangle = (F_+F_{05} + F_+)|\mu\rangle = F_+F_{05}|\mu\rangle + F_+|\mu\rangle = F_+\mu|\mu\rangle + F_+|\mu\rangle = (\mu+1)F_+|\mu\rangle\,. \tag{10}$$

If $F_-|\mu\rangle$ is not the zero vector, then it is an eigenvector of F_{05} with eigenvalue $\mu-1$,

$$F_{05}F_-|\mu\rangle = (F_-F_{05} - F_-)|\mu\rangle = F_-F_{05}|\mu\rangle - F_-|\mu\rangle = F_-\mu|\mu\rangle - F_-|\mu\rangle = (\mu-1)F_-|\mu\rangle\,. \tag{11}$$

For $n \in \{0,1,2,\ldots\}$ repeated applications of the operators, if $F_+^n|\mu\rangle$ is not the zero vector, then it is an eigenvector of F_{05} with eigenvalue $\mu+n$, and, if $F_-^n|\mu\rangle$ is not the zero vector, then it is an eigenvector of F_{05} with eigenvalue $\mu-n$.

We restrict consideration to representations with a scalar product that fulfills

$$(F_{05}|\psi\rangle,|\phi\rangle) = (|\psi\rangle,F_{05}|\phi\rangle)\,, \quad (F_{01}|\psi\rangle,|\phi\rangle) = (|\psi\rangle,F_{01}|\phi\rangle)\,, \quad (F_{15}|\psi\rangle,|\phi\rangle) = (|\psi\rangle,F_{15}|\phi\rangle)\,, \quad \text{for all } |\phi\rangle \text{ and } |\psi\rangle, \tag{12}$$

and, consequently,[1]

$$\begin{aligned} (F_-|\psi\rangle,|\phi\rangle) &= \Big(\tfrac{1}{\sqrt{2}}(F_{01} - iF_{15})|\psi\rangle,|\phi\rangle\Big) = \tfrac{1}{\sqrt{2}}(F_{01}|\psi\rangle,|\phi\rangle) + \tfrac{i}{\sqrt{2}}(F_{15}|\psi\rangle,|\phi\rangle) = \tfrac{1}{\sqrt{2}}(|\psi\rangle,F_{01}|\phi\rangle) + \tfrac{i}{\sqrt{2}}(|\psi\rangle,F_{15}|\phi\rangle) \\ &= \Big(|\psi\rangle,\tfrac{1}{\sqrt{2}}(F_{01}|\phi\rangle + iF_{15})|\phi\rangle\Big) = (|\psi\rangle,F_+|\phi\rangle)\,, \qquad \text{for all } |\phi\rangle \text{ and } |\psi\rangle, \end{aligned} \tag{13}$$

where the first and fifth equalities use Eq. (4), the second equality uses anti-linearity of the left argument of the scalar product, the third equality uses Eq. (12), and the fourth equality uses linearity of the right argument of the scalar product. A representation that fulfills Eq. (12) is called *hermitian*.

---

[1] Operators, such as F_+ and F_-, which fulfill Eq. (13) are called *adjoints* of each other, often denoted by $F_-^\dagger = F_+$ and $F_+^\dagger = F_-$. Operators, such as F_{05}, F_{01}, and F_{15}, which fulfill Eq. (12) are *self-adjoint*, $F_{05}^\dagger = F_{05}$, $F_{01}^\dagger = F_{01}$, and $F_{15}^\dagger = F_{15}$.

For a hermitian representation, use of $|\psi\rangle = |\mu\rangle$ and $|\phi\rangle = |\mu\rangle$ in Eq. (12) shows that eigenvalues of F_{05} are real,

$$\overline{\mu}\,\langle\mu|\mu\rangle = (\mu|\mu\rangle, |\mu\rangle) = (F_{05}|\mu\rangle, |\mu\rangle) = (|\mu\rangle, F_{05}|\mu\rangle) = (|\mu\rangle, \mu|\mu\rangle) = \mu\,\langle\mu|\mu\rangle \qquad \text{or} \qquad \overline{\mu} = \mu. \tag{14}$$

We further restrict consideration to representations with a lowest eigenvalue $\mu_{\rm min}$ of F_{05}, called *lowest-weight* hermitian representations, and representations with a highest eigenvalue $\mu_{\rm max}$ of F_{05}, called *highest-weight* hermitian representations.

Let $|\mu_{\rm min}\rangle$ be an eigenvector of F_{05} with lowest eigenvalue $\mu_{\rm min}$. The vector $F_-|\mu_{\rm min}\rangle$ must then be the zero vector in order not to be an eigenvector of F_{05} with eigenvalue lower than $\mu_{\rm min}$. All vectors obtained by acting with operators $\mathcal{E}$ in the enveloping algebra $\mathcal{E}(\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}})$ on the vector $|\mu_{\rm min}\rangle$ are eigenvectors of the operator of Eq. (7) with eigenvalue $\mu_{\rm min}(\mu_{\rm min}-1)$,

$$\begin{aligned}(F_{05}F_{05}-F_{05}-2F_+F_-)\,\mathcal{E}|\mu_{\rm min}\rangle &= \mathcal{E}(F_{05}F_{05}-F_{05}-2F_+F_-)|\mu_{\rm min}\rangle \\ &= \mathcal{E}\mu_{\rm min}(\mu_{\rm min}-1)|\mu_{\rm min}\rangle = \mu_{\rm min}(\mu_{\rm min}-1)\,\mathcal{E}\,|\mu_{\rm min}\rangle\,.\end{aligned} \tag{15}$$

Actions of the operators F_+ and F_- on the vector $|\mu\rangle$ may be expressed as

$$F_+|\mu\rangle = u(\mu)|\mu+1\rangle\,, \qquad\qquad F_-|\mu\rangle = d(\mu)|\mu-1\rangle\,, \tag{16}$$

where $|\mu+1\rangle$ is an eigenvector of F_{05} with eigenvalue $\mu+1$, $|\mu-1\rangle$ is an eigenvector of F_{05} with eigenvalue $\mu-1$, and $u(\mu)$ and $d(\mu)$ are functions of μ. A condition on the functions $u(\mu)$ and $d(\mu)$ is obtained from the action of the operator of Eq. (7) on the vector $|\mu\rangle$ (which equals $\mathcal{E}|\mu_{\rm min}\rangle$ for an appropriate operator $\mathcal{E}$ in the enveloping algebra),

$$\mu_{\rm min}(\mu_{\rm min}-1)|\mu\rangle = (F_{05}F_{05}-F_{05}-2F_+F_-)|\mu\rangle = [\mu(\mu-1)-2u(\mu-1)\,d(\mu)]|\mu\rangle\,, \tag{17}$$

or

$$u(\mu-1)\,d(\mu) = [\mu(\mu-1)-\mu_{\rm min}(\mu_{\rm min}-1)]/2 = (\mu+\mu_{\rm min}-1)(\mu-\mu_{\rm min})/2. \tag{18}$$

Another condition on the functions $u(\mu)$ and $d(\mu)$ is obtained by use of $|\psi\rangle = |\mu\rangle$ and $|\phi\rangle = |\mu-1\rangle$ in Eq. (13),

$$\begin{aligned}\overline{d(\mu)}\,\langle\mu-1|\mu-1\rangle &= (d(\mu)|\mu-1\rangle, |\mu-1\rangle) = (F_-|\mu\rangle, |\mu-1\rangle) \\ &= (|\mu\rangle, F_+|\mu-1\rangle) = (|\mu\rangle, u(\mu-1)|\mu\rangle) = u(\mu-1)\,\langle\mu|\mu\rangle\,.\end{aligned} \tag{19}$$

Elimination of $u(\mu-1)$ from Eqs. (18) and (19) gives

$$|d(\mu)|^2\,\langle\mu-1|\mu-1\rangle = (\mu+\mu_{\rm min}-1)(\mu-\mu_{\rm min})/2\,\langle\mu|\mu\rangle\,. \tag{20}$$

Since $|d(\mu)|^2$ and the squared lengths $\langle\mu|\mu\rangle$ and $\langle\mu-1|\mu-1\rangle$ of vectors $|\mu\rangle$ and $|\mu-1\rangle$ are real and non-negative, either $(\mu+\mu_{\rm min}-1)(\mu-\mu_{\rm min})/2$ is real and non-negative or $|\mu\rangle$ and $F_-|\mu\rangle = d(\mu)|\mu-1\rangle$ are the zero vector. For $d(\mu) = |d(\mu)|e^{i\theta_\mu}$ with any phase factor $e^{i\theta_\mu}$,

$$d(\mu)\sqrt{\langle\mu-1|\mu-1\rangle} = \sqrt{(\mu+\mu_{\rm min}-1)(\mu-\mu_{\rm min})/2}\;e^{i\theta_\mu}\sqrt{\langle\mu|\mu\rangle}. \tag{21}$$

Elimination of $d(\mu)$ from Eqs. (18) and (21) gives

$$u(\mu-1)\sqrt{\langle\mu|\mu\rangle} = \sqrt{(\mu+\mu_{\rm min}-1)(\mu-\mu_{\rm min})/2}\;e^{-i\theta_\mu}\sqrt{\langle\mu-1|\mu-1\rangle}. \tag{22}$$

or, with substitution of $\mu+1$ for μ,

$$u(\mu)\sqrt{\langle\mu+1|\mu+1\rangle} = \sqrt{(\mu+\mu_{\rm min})(\mu-\mu_{\rm min}+1)/2}\;e^{-i\theta_{\mu+1}}\sqrt{\langle\mu|\mu\rangle}. \tag{23}$$

Elimination of $u(\mu)$ and $d(\mu)$ from Eqs. (16), (21), and (23) gives

$$F_+|\mu\rangle/\sqrt{\langle\mu|\mu\rangle} = \sqrt{(\mu+\mu_{\rm min})(\mu-\mu_{\rm min}+1)/2}\;e^{-i\theta_{\mu+1}}\,|\mu+1\rangle/\sqrt{\langle\mu+1|\mu+1\rangle}, \tag{24}$$

$$F_-|\mu\rangle/\sqrt{\langle\mu|\mu\rangle} = \sqrt{(\mu+\mu_{\rm min}-1)(\mu-\mu_{\rm min})/2}\;e^{i\theta_\mu}\,|\mu-1\rangle/\sqrt{\langle\mu-1|\mu-1\rangle}. \tag{25}$$

Substitution of $\mu = \mu_{\rm min}+n$ in Eqs. (9), (24), and (25) gives

$$F_{05}|\mu_{\rm min}+n\rangle = (\mu_{\rm min}+n)|\mu_{\rm min}+n\rangle\,, \tag{26}$$

$$F_+\frac{|\mu_{\rm min}+n\rangle}{\sqrt{\langle\mu_{\rm min}+n|\mu_{\rm min}+n\rangle}} = \sqrt{(2\mu_{\rm min}+n)(n+1)/2}\;e^{-i\theta_{\mu_{\rm min}+n+1}}\frac{|\mu_{\rm min}+n+1\rangle}{\sqrt{\langle\mu_{\rm min}+n+1|\mu_{\rm min}+n+1\rangle}}, \tag{27}$$

$$F_-\frac{|\mu_{\rm min}+n\rangle}{\sqrt{\langle\mu_{\rm min}+n|\mu_{\rm min}+n\rangle}} = \sqrt{(2\mu_{\rm min}+n-1)\,n/2}\;e^{i\theta_{\mu_{\rm min}+n}}\frac{|\mu_{\rm min}+n-1\rangle}{\sqrt{\langle\mu_{\rm min}+n-1|\mu_{\rm min}+n-1\rangle}}. \tag{28}$$

Actions of the operators F_{05}, F_+, and F_- and of the operator of Eq. (7) on the unit length vectors

$$|n\rangle = e^{i\alpha_n}|\mu_{\rm min}+n\rangle/\sqrt{\langle\mu_{\rm min}+n\,|\mu_{\rm min}+n\rangle}, \qquad \text{where } \alpha_n = \alpha_{n-1}-\theta_{\mu_{\rm min}+n} \text{ (modulo } 2\pi), \tag{29}$$

are given by

$$F_{05}|n\rangle = (\mu_{\rm min}+n)|n\rangle\,, \tag{30}$$

$$F_+|n\rangle = \sqrt{(2\mu_{\rm min}+n)(n+1)/2}\,|n+1\rangle\,, \tag{31}$$

$$F_-|n\rangle = \sqrt{(2\mu_{\rm min}+n-1)\,n/2}\,|n-1\rangle\,, \tag{32}$$

$$(F_{05}F_{05}-F_{01}F_{01}-F_{15}F_{15})|n\rangle = \mu_{\rm min}(\mu_{\rm min}-1)|n\rangle\,. \tag{33}$$

Use of Eqs. (12) and (14) shows that, if $n' \neq n$, then the unit length vectors $|n'\rangle$ and $|n\rangle$ are orthogonal,

$$
\begin{aligned}
(\mu_{\rm min}+n')\,\langle n'|n\rangle &= \overline{(\mu_{\rm min}+n')}\,\langle n'|n\rangle = ((\mu_{\rm min}+n')|n'\rangle, |n\rangle) \\
&= (F_{05}|n'\rangle, |n\rangle) = (|n'\rangle, F_{05}|n\rangle) = (|n'\rangle, (\mu_{\rm min}+n)|n\rangle) = (\mu_{\rm min}+n)\,\langle n'|n\rangle\,,
\end{aligned} \tag{34}
$$

or

$$
(n'-n)\,\langle n'|n\rangle = 0, \qquad \text{or, if } n' \neq n, \text{ then} \qquad \langle n'|n\rangle = 0. \tag{35}
$$

For all lowest-weight hermitian irreducible ray representations (lowest-weight *hermirreps*) $\mathcal{D}_{F_{05},F_{01},F_{15}}(\mu_{\rm min})$ of the Lie algebra of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ the lowest value of n is 0. If $0 < \mu_{\rm min} < \infty$, then, for $n \in \{0,1,2,\ldots\}$, $F_+|n\rangle$ is never the zero vector, repeated applications of F_+ reach all higher values of n, and the vector space $\mathcal{H}_{F_{05},F_{01},F_{15}}(\mu_{\rm min})$ of the lowest-weight hermirreps is ∞-dimensional. If $\mu_{\rm min} = 0$, then, for $n = 0$, $F_+|0\rangle$ is the zero vector, application of F_+ does not reach $n = 1$, and $\mathcal{H}_{F_{05},F_{01},F_{15}}(\mu_{\rm min})$ is a 1-dimensional vector space spanned by the (non-zero) unit length vector $|0\rangle$. If $-\infty < \mu_{\rm min} < 0$, then no lowest-weight hermirrep exists since

$$
(F_+|0\rangle, F_+|0\rangle) = (|0\rangle, F_-F_+|0\rangle) = (|0\rangle, F_-\sqrt{\mu_{\rm min}}|1\rangle) = (|0\rangle, \sqrt{\mu_{\rm min}}F_-|1\rangle) = (|0\rangle, \mu_{\rm min}|0\rangle) = \mu_{\rm min}\,\langle 0|0\rangle \tag{36}
$$

is not satisfied for vectors $|0\rangle$ and $F_+|0\rangle$ with squared lengths $\langle 0|0\rangle$ and $(F_+|0\rangle, F_+|0\rangle)$ greater than zero; $|0\rangle$ and $F_+|0\rangle = \sqrt{\mu_{\rm min}}\,|1\rangle$ are the zero vector.

Lowest-weight hermirreps $\mathcal{D}_{F_{05},F_{01},F_{15}}(\mu_{1\rm min})$ of the Lie algebra of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ are characterized by their lowest eigenvalue $\mu_{\rm min}$ of F_{05} and exist if and only if

$$
0 \leq \mu_{\rm min} < \infty. \tag{37}
$$

The vector space $\mathcal{H}_{F_{05},F_{01},F_{15}}(\mu_{\rm min})$ of a lowest-weight hermirrep $\mathcal{D}_{F_{05},F_{01},F_{15}}(\mu_{\rm min})$ is spanned by a basis of eigenvectors $|n\rangle$ of F_{05} for which

$$
n = 0 \quad \text{if} \quad \mu_{\rm min} = 0 \qquad \text{and} \qquad n \in \{0,1,2,\ldots\} \quad \text{if} \quad 0 < \mu_{\rm min} < \infty \tag{38}
$$

and on which the basis operators of Eq. (4) have actions given in Eqs. (30), (31), and (32) and the operator of Eq. (7) has action given in Eq. (33). The vector space $\mathcal{H}_{F_{05},F_{01},F_{15}}(\mu_{\rm min})$ has dimension 1 if $\mu_{\rm min} = 0$ and dimension ∞ if $0 < \mu_{\rm min} < \infty$. The basis vectors $|n\rangle$ are unit length and orthogonal,

$$
\langle n'|n\rangle = \delta_{n'n} = \begin{cases} 1, & \text{if } n' = n, \\ 0, & \text{if } n' \neq n. \end{cases} \tag{39}
$$

Highest-weight hermirreps $\mathcal{D}_{F_{05},F_{01},F_{15}}(\mu_{\rm max})$ of the Lie algebra of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ are characterized by their highest eigenvalue $\mu_{\rm max}$ of F_{05} and exist if and only if

$$
-\infty < \mu_{\rm max} \leq 0. \tag{40}
$$

The vector space $\mathcal{H}_{F_{05},F_{01},F_{15}}(\mu_{\rm max})$ of a highest-weight hermirrep $\mathcal{D}_{F_{05},F_{01},F_{15}}(\mu_{\rm max})$ is spanned by a basis of eigenvectors $|n\rangle$ of F_{05} for which

$$
n \in \{0,-1,-2,\ldots\} \quad \text{if} \quad -\infty < \mu_{\rm max} < 0 \qquad \text{and} \qquad n = 0 \quad \text{if} \quad \mu_{\rm max} = 0 \tag{41}
$$

and on which the basis operators of Eq. (4) have actions

$$
F_{05}\,|n\rangle = (\mu_{\rm max}+n)|n\rangle\,, \tag{42}
$$

$$
F_-\,|n\rangle = \sqrt{(2\mu_{\rm max}+n)(n-1)/2}\,|n-1\rangle\,, \tag{43}
$$

$$
F_+\,|n\rangle = \sqrt{(2\mu_{\rm max}+n+1)\,n/2}\,|n+1\rangle \tag{44}
$$

and the operator of Eq. (7) has action

$$
(F_{05}F_{05}-F_{01}F_{01}-F_{15}F_{15})\,|n\rangle = \mu_{\rm max}\,(\mu_{\rm max}+1)|n\rangle\,. \tag{45}
$$

The vector space $\mathcal{H}_{F_{05},F_{01},F_{15}}(\mu_{\rm max})$ has dimension ∞ if $-\infty < \mu_{\rm max} < 0$ and 1 if $\mu_{\rm max} = 0$. The basis vectors $|n\rangle$ are unit length and orthogonal,

$$
\langle n'|n\rangle = \delta_{n'n} = \begin{cases} 1, & \text{if } n' = n, \\ 0, & \text{if } n' \neq n. \end{cases} \tag{46}
$$

A lowest-weight hermirrep space $\mathcal{H}_{F_{05},F_{01},F_{15}}(0<\mu_{\rm min}<\infty)$ or highest-weight hermirrep space $\mathcal{H}_{F_{05},F_{01},F_{15}}(-\infty<\mu_{\rm max}<0)$ of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ reduces into an infinite direct sum of hermirrep spaces $\mathcal{H}_{F_{05}}(\mu)$ of the $\mathrm{SO}(2)_{F_{05}}$ subalgebra,[2]

$$
\mathcal{H}_{F_{05},F_{01},F_{15}}(0<\mu_{\rm min}<\infty) = \sum_{\mu=\mu_{\rm min},\mu_{\rm min}+1,\mu_{\rm min}+2,\cdots}^{\infty} \oplus\mathcal{H}_{F_{05}}(\mu) = \sum_{n=0,1,2,\cdots}^{\infty} \oplus\mathcal{H}_{F_{05}}(\mu_{\rm min}+n), \tag{47}
$$

$$
\mathcal{H}_{F_{05},F_{01},F_{15}}(-\infty<\mu_{\rm max}<0) = \sum_{\mu=\mu_{\rm max},\mu_{\rm max}-1,\mu_{\rm max}-2,\cdots}^{-\infty} \oplus\mathcal{H}_{F_{05}}(\mu) = \sum_{n=0,-1,-2,\cdots}^{-\infty} \oplus\mathcal{H}_{F_{05}}(\mu_{\rm max}+n). \tag{48}
$$

The only hermirrep of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ that remains irreducible with respect to $\mathrm{SO}(2)_{F_{05}}$ is $\mathcal{D}_{F_{05},F_{01},F_{15}}(0)$.

[2] Hermirreps $\mathcal{D}_{F_{05}}(\mu)$ of $\mathrm{SO}(2)_{F_{05}}$ are characterized by the eigenvalue μ of F_{05} and exist if and only if $\mu \in \Re$; the vector space $\mathcal{H}_{F_{05}}(\mu)$ of a hermirrep $\mathcal{D}_{F_{05}}(\mu)$ of $\mathrm{SO}(2)_{F_{05}}$ has dimension 1.

In the limit $\mu_{\min}\to 0$ through the continuous sequence of hermirreps $\mathcal{D}_{F_{05},F_{01},F_{15}}(0<\mu_{\min}<\infty)$, those matrix elements of F_+ and F_- that connect the subspace $\mathcal{H}_{F_{05}}(\mu_{\min})$ of $\mathcal{H}_{F_{05},F_{01},F_{15}}(0<\mu_{\min}<\infty)$ for which $n=0$ to the remaining subspace become zero and the hermirrep splits into the direct sum of two hermirreps:

$$\lim_{\mu_{\min}\to 0}\mathcal{D}_{F_{05},F_{01},F_{15}}(0<\mu_{\min}<\infty)=\mathcal{D}_{F_{05},F_{01},F_{15}}(\mu_{\min}=0)+\mathcal{D}_{F_{05},F_{01},F_{15}}(\mu_{\min}+1=1)\,. \tag{49}$$

In the limit $\mu_{\max}\to 0$ through the continuous sequence of hermirreps $\mathcal{D}_{F_{05},F_{01},F_{15}}(-\infty<\mu_{\max}<0)$, those matrix elements of F_+ and F_- that connect the subspace $\mathcal{H}_{F_{05}}(\mu_{\max})$ of $\mathcal{H}_{F_{05},F_{01},F_{15}}(-\infty<\mu_{\max}<0)$ for which $n=0$ to the remaining subspace become zero and the hermirrep splits into the direct sum of two hermirreps:

$$\lim_{\mu_{\max}\to 0}\mathcal{D}_{F_{05},F_{01},F_{15}}(-\infty<\mu_{\max}<0)=\mathcal{D}_{F_{05},F_{01},F_{15}}(\mu_{\max}=0)+\mathcal{D}_{F_{05},F_{01},F_{15}}(\mu_{\max}-1=-1)\,. \tag{50}$$

These are examples of *representation splitting*.

In the limit $\mu_{\min}\to\infty$ through the continuous sequence of hermirreps $\mathcal{D}_{F_{05},F_{01},F_{15}}(0<\mu_{\min}<\infty)$, the operators

$$I=\lim_{\mu_{\min}\to\infty}F_{05}/\mu_{\min},\qquad a=\lim_{\mu_{\min}\to\infty}F_-/\sqrt{\mu_{\min}},\qquad a^\dagger=\lim_{\mu_{\min}\to\infty}F_+/\sqrt{\mu_{\min}} \tag{51}$$

have the commutation relations

$$\begin{array}{ccc}[I,I]=0, & [a,a]=0, & [a^\dagger,a^\dagger]=0,\\ [a,a^\dagger]=I, & [I,a^\dagger]=0, & [I,a]=0,\\ [a^\dagger,a]=-I, & [a^\dagger,I]=0, & [a,I]=0\end{array} \tag{52}$$

of the Heisenberg-Weyl Lie algebra $\mathrm{HW}_{I,a,a^\dagger}(1)$ and have the usual one-dimensional oscillator actions:

$$I\,|n\rangle=|n\rangle\,,\qquad a\,|n\rangle=\sqrt{n}\,|n{-}1\rangle\,,\qquad a^\dagger\,|n\rangle=\sqrt{n{+}1}\,|n{+}1\rangle\,. \tag{53}$$

In the limit $\mu_{\max}\to-\infty$ through the continuous sequence of hermirreps $\mathcal{D}_{F_{05},F_{01},F_{15}}(-\infty<\mu_{\max}<0)$, the operators

$$I=\lim_{\mu_{\max}\to-\infty}F_{05}/\mu_{\max},\qquad a=\lim_{\mu_{\max}\to-\infty}F_+/\sqrt{-\mu_{\max}},\qquad a^\dagger=\lim_{\mu_{\max}\to-\infty}F_-/\sqrt{-\mu_{\max}} \tag{54}$$

also have the commutation relations of Eq. (52) and have the usual one-dimensional oscillator actions:

$$I\,|n\rangle=|n\rangle\,,\qquad a\,|n\rangle=\sqrt{-n}\,|n{+}1\rangle\,,\qquad a^\dagger\,|n\rangle=\sqrt{-n{+}1}\,|n{-}1\rangle\,; \tag{55}$$

that these are the usual actions of the one-dimensional oscillator follows from the change of variable $n=-m$, with $n\in\{0,-1,-2,\ldots\}$ and $m\in\{0,1,2,\ldots\}$,

$$I\,|{-m}\rangle=|{-m}\rangle\,,\qquad a\,|{-m}\rangle=\sqrt{m}\,|{-m}{+}1\rangle\,,\qquad a^\dagger\,|{-m}\rangle=\sqrt{m{+}1}\,|{-m}{-}1\rangle\,, \tag{56}$$

and from use of $|{-\mathrm{arg}}\rangle=|\mathrm{arg}\}$ to denote sign changes of the entire arguments of the basis vectors,

$$I\,|m\}=|m\}\,,\qquad a\,|m\}=\sqrt{m}\,|m{-}1\}\,,\qquad a^\dagger\,|m\}=\sqrt{m{+}1}\,|m{+}1\}\,. \tag{57}$$

These are examples of *contraction* through the continuous sequences of hermirreps $\mathcal{D}_{F_{05},F_{01},F_{15}}(0<\mu_{\min}<\infty)$ and $\mathcal{D}_{F_{05},F_{01},F_{15}}(-\infty<\mu_{\max}<0)$ of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ to obtain the hermirrep $\mathcal{D}_{I,a,a^\dagger}$ of $\mathrm{HW}_{I,a,a^\dagger}(1)$. The vector space of all of these hermirreps is $\mathcal{H}_{F_{05},F_{01},F_{15}}(0<\mu_{\min}<\infty)=\mathcal{H}_{I,a,a^\dagger}=\mathcal{H}_{F_{05},F_{01},F_{15}}(-\infty<\mu_{\max}<0)$.

For all lowest-weight hermirreps $\mathcal{D}_{F_{05},F_{01},F_{15}}(0\le\mu_{\min}<\infty)$ of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$, the operators of Eq. (4) are given as functions of the operators of Eq. (51) by

$$F_+=a^\dagger\sqrt{\mu_{\min}I+\tfrac{1}{2}a^\dagger a},\qquad F_{05}=\mu_{\min}I+a^\dagger a,\qquad F_-=\sqrt{\mu_{\min}I+\tfrac{1}{2}a^\dagger a}\;a \tag{58}$$

and the operators of Eq. (1) are given as functions of the operators of Eq. (51) by

$$\begin{aligned}F_{01}&=\tfrac{1}{\sqrt{2}}\Big(\sqrt{\mu_{\min}I+\tfrac{1}{2}a^\dagger a}\;a+a^\dagger\sqrt{\mu_{\min}I+\tfrac{1}{2}a^\dagger a}\Big),\quad F_{05}=\mu_{\min}I+a^\dagger a,\\ F_{15}&=\tfrac{i}{\sqrt{2}}\Big(\sqrt{\mu_{\min}I+\tfrac{1}{2}a^\dagger a}\;a-a^\dagger\sqrt{\mu_{\min}I+\tfrac{1}{2}a^\dagger a}\Big).\end{aligned} \tag{59}$$

For all highest-weight hermirreps $\mathcal{D}_{F_{05},F_{01},F_{15}}(-\infty<\mu_{\max}\le 0)$ of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$, the operators of Eq. (4) are given as functions of the operators of Eq. (54) by

$$F_+=\sqrt{-\mu_{\max}I+\tfrac{1}{2}a^\dagger a}\;a,\qquad F_{05}=\mu_{\max}I-a^\dagger a,\qquad F_-=a^\dagger\sqrt{-\mu_{\max}I+\tfrac{1}{2}a^\dagger a} \tag{60}$$

and the operators of Eq. (1) are given as functions of the operators of Eq. (54) by

$$\begin{aligned}F_{01}&=\tfrac{1}{\sqrt{2}}\Big(a^\dagger\sqrt{-\mu_{\max}I+\tfrac{1}{2}a^\dagger a}+\sqrt{-\mu_{\max}I+\tfrac{1}{2}a^\dagger a}\;a\Big),\quad F_{05}=\mu_{\max}I-a^\dagger a,\\ F_{15}&=\tfrac{i}{\sqrt{2}}\Big(a^\dagger\sqrt{-\mu_{\max}I+\tfrac{1}{2}a^\dagger a}-\sqrt{-\mu_{\max}I+\tfrac{1}{2}a^\dagger a}\;a\Big).\end{aligned} \tag{61}$$

These are examples of *realization* of a basis of operators in the Lie algebra of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ as functions of a basis of operators in the Lie algebra of $\mathrm{HW}_{I,a_1,a_1^\dagger}(1)$.