Lowest-weight and highest-weight hermirreps of SO(2,2), representation splitting, contraction to SO(2)⋉HW(2), and realization of SO(2,2) basis operators as functions of SO(2)⋉HW(2) basis operators

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The (complexified) Lie algebra of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$ has a basis of six operators,

$$\begin{matrix} F_{01}, & F_{02}, & F_{05}, \\ & F_{12}, & F_{15}, \\ & & F_{25}, \end{matrix} \tag{1}$$

that satisfy the commutation relations

$$[F_{ab},F_{cd}] \equiv F_{ab}F_{cd}-F_{cd}F_{ab} = -i\,(\eta_{ac}F_{bd}+\eta_{bd}F_{ac}-\eta_{ad}F_{bc}-\eta_{bc}F_{ad}), \qquad \text{for } a,b\in\{0,1,2,5\}, \tag{2}$$

where $F_{ab}=-F_{ba}$, $\eta_{00}=-\eta_{11}=-\eta_{22}=\eta_{55}=1$, and $\eta_{ab}=0$ for $a\neq b$. A commutator of Eq. (2) is non-zero if and only if only one of the indices of F_{ab} equals one of the indices of F_{cd}. All 36 commutators are

$$\begin{array}{llllll}
[F_{01},F_{01}]=0, & [F_{01},F_{02}]=-iF_{12}, & [F_{01},F_{12}]=-iF_{02}, & [F_{02},F_{12}]=iF_{01}, & [F_{05},F_{15}]=-iF_{01}, & [F_{01},F_{25}]=0,\\
[F_{02},F_{02}]=0, & [F_{02},F_{01}]=iF_{12}, & [F_{12},F_{01}]=iF_{02}, & [F_{12},F_{02}]=-iF_{01}, & [F_{15},F_{05}]=iF_{01}, & [F_{25},F_{01}]=0,\\
[F_{05},F_{05}]=0, & [F_{01},F_{05}]=-iF_{15}, & [F_{01},F_{15}]=-iF_{05}, & [F_{02},F_{25}]=-iF_{05}, & [F_{05},F_{25}]=-iF_{02}, & [F_{02},F_{15}]=0,\\
[F_{12},F_{12}]=0, & [F_{05},F_{01}]=iF_{15}, & [F_{15},F_{01}]=iF_{05}, & [F_{25},F_{02}]=iF_{05}, & [F_{25},F_{05}]=iF_{02}, & [F_{15},F_{02}]=0,\\
[F_{15},F_{15}]=0, & [F_{02},F_{05}]=-iF_{25}, & [F_{12},F_{15}]=iF_{25}, & [F_{12},F_{25}]=-iF_{15}, & [F_{15},F_{25}]=-iF_{12}, & [F_{05},F_{12}]=0,\\
[F_{25},F_{25}]=0, & [F_{05},F_{02}]=iF_{25}, & [F_{15},F_{12}]=-iF_{25}, & [F_{25},F_{12}]=iF_{15}, & [F_{25},F_{15}]=iF_{12}, & [F_{12},F_{05}]=0.
\end{array} \tag{3}$$

A basis in the Lie algebra of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$ that consists of operators which raise by 1, lower by 1, or leave unchanged the eigenvalues of F_{05} and F_{12} is given by the operators

$$\begin{array}{lcl}
F_{+-0}=-\tfrac{1}{2}(F_{01}+F_{25}-iF_{02}+iF_{15}), & & F_{++0}=\ \tfrac{1}{2}(F_{01}-F_{25}+iF_{02}+iF_{15}),\\
 & F_{05},\quad F_{12}, & \\
F_{--0}=\ \tfrac{1}{2}(F_{01}-F_{25}-iF_{02}-iF_{15}), & & F_{-+0}=-\tfrac{1}{2}(F_{01}+F_{25}+iF_{02}-iF_{15}).
\end{array} \tag{4}$$

The basis of Eq. (1) is given in terms of the basis of Eq. (4) by

$$\begin{array}{lll}
F_{01}=\tfrac{1}{2}(F_{++0}+F_{--0}-F_{+-0}-F_{-+0}), & F_{02}=-\tfrac{i}{2}(F_{++0}-F_{--0}+F_{+-0}-F_{-+0}), & F_{05},\\
 & F_{12}, & F_{15}=-\tfrac{i}{2}(F_{++0}-F_{--0}-F_{+-0}+F_{-+0}),\\
 & & F_{25}=-\tfrac{1}{2}(F_{++0}+F_{--0}+F_{+-0}+F_{-+0}).
\end{array} \tag{5}$$

The commutation relations of the operators of Eq. (4) are

$$\begin{array}{l}
[F_{05},F_{05}]=0,\quad [F_{12},F_{12}]=0,\quad [F_{++0},F_{++0}]=0,\quad [F_{--0},F_{--0}]=0,\quad [F_{+-0},F_{+-0}]=0,\quad [F_{-+0},F_{-+0}]=0,\\
[F_{05},F_{12}]=0,\quad [F_{++0},F_{--0}]=-F_{05}-F_{12},\quad [F_{+-0},F_{-+0}]=-F_{05}+F_{12},\\
[F_{12},F_{05}]=0,\quad [F_{--0},F_{++0}]=F_{05}+F_{12},\quad [F_{-+0},F_{+-0}]=F_{05}-F_{12},\\
[F_{++0},F_{+-0}]=0,\quad [F_{++0},F_{-+0}]=0,\quad [F_{--0},F_{+-0}]=0,\quad [F_{--0},F_{-+0}]=0,\\
[F_{+-0},F_{++0}]=0,\quad [F_{-+0},F_{++0}]=0,\quad [F_{+-0},F_{--0}]=0,\quad [F_{-+0},F_{--0}]=0,\\
[F_{05},F_{++0}]=F_{++0},\quad [F_{05},F_{--0}]=-F_{--0},\quad [F_{05},F_{+-0}]=F_{+-0},\quad [F_{05},F_{-+0}]=-F_{-+0},\\
[F_{++0},F_{05}]=-F_{++0},\quad [F_{--0},F_{05}]=F_{--0},\quad [F_{+-0},F_{05}]=-F_{+-0},\quad [F_{-+0},F_{05}]=F_{-+0},\\
[F_{12},F_{++0}]=F_{++0},\quad [F_{12},F_{--0}]=-F_{--0},\quad [F_{12},F_{+-0}]=-F_{+-0},\quad [F_{12},F_{-+0}]=F_{-+0},\\
[F_{++0},F_{12}]=-F_{++0},\quad [F_{--0},F_{12}]=F_{--0},\quad [F_{+-0},F_{12}]=F_{+-0},\quad [F_{-+0},F_{12}]=-F_{-+0}.
\end{array} \tag{6}$$

The second-order operators

$$\begin{aligned}
&F_{05}F_{05}+F_{12}F_{12}-F_{01}F_{01}-F_{02}F_{02}-F_{15}F_{15}-F_{25}F_{25}\\
&=F_{05}F_{05}+F_{12}F_{12}-F_{++0}F_{--0}-F_{--0}F_{++0}-F_{+-0}F_{-+0}-F_{-+0}F_{+-0}\\
&=F_{05}F_{05}+F_{12}F_{12}-2F_{++0}F_{--0}-2F_{+-0}F_{-+0}-2F_{05}\\
&=F_{05}F_{05}+F_{12}F_{12}-2F_{--0}F_{++0}-2F_{-+0}F_{+-0}+2F_{05}\\
&=\tfrac{1}{2}(F_{05}+F_{12})^2-F_{++0}F_{--0}-F_{--0}F_{++0}+\tfrac{1}{2}(F_{05}-F_{12})^2-F_{+-0}F_{-+0}-F_{-+0}F_{+-0},
\end{aligned} \tag{7}$$

$$\begin{aligned}
W_{34}&=F_{05}F_{12}-F_{02}F_{15}+F_{01}F_{25}\\
&=F_{05}F_{12}-\tfrac{1}{2}F_{++0}F_{--0}-\tfrac{1}{2}F_{--0}F_{++0}+\tfrac{1}{2}F_{+-0}F_{-+0}+\tfrac{1}{2}F_{-+0}F_{+-0}\\
&=F_{05}F_{12}-F_{++0}F_{--0}+F_{+-0}F_{-+0}-F_{12}\\
&=F_{05}F_{12}-F_{--0}F_{++0}+F_{-+0}F_{+-0}+F_{12}\\
&=\tfrac{1}{4}(F_{05}+F_{12})^2-\tfrac{1}{2}F_{++0}F_{--0}-\tfrac{1}{2}F_{--0}F_{++0}-\tfrac{1}{4}(F_{05}-F_{12})^2+\tfrac{1}{2}F_{+-0}F_{-+0}+\tfrac{1}{2}F_{-+0}F_{+-0}
\end{aligned} \tag{8}$$

commute with all operators

$$\mathcal{E}=cI+\sum_{a,b}c^{ab}F_{ab}+\sum_{a,b,c,d}c^{abcd}F_{ab}F_{cd}+\sum_{a,b,c,d,e,f}c^{abcdef}F_{ab}F_{cd}F_{ef}+\cdots,\qquad c,c^{ab},c^{abcd},c^{abcdef},\ldots\in\mathcal{C}, \tag{9}$$

in the enveloping algebra $\mathcal{E}(\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}})$ of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$.

Addition and subtraction of twice Eq. (8) to and from Eq. (7) gives the respective operators

$$(F_{05}+F_{12})(F_{05}+F_{12}-2I)-4F_{++0}F_{--0} = (F_{05}+F_{12})(F_{05}+F_{12}+2I)-4F_{--0}F_{++0}, \tag{10}$$

$$(F_{05}-F_{12})(F_{05}-F_{12}-2I)-4F_{+-0}F_{-+0} = (F_{05}-F_{12})(F_{05}-F_{12}+2I)-4F_{-+0}F_{+-0}, \tag{11}$$

which also commute with all operators $\mathcal{E}$ in the enveloping algebra $\mathcal{E}(\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}})$.

Let $|\mu\ f_3\rangle$ be an eigenvector of F_{05} with eigenvalue μ and an eigenvector of F_{12} with eigenvalue f_3,

$$F_{05}|\mu\ f_3\rangle = \mu|\mu\ f_3\rangle\,, \tag{12}$$

$$F_{12}|\mu\ f_3\rangle = f_3|\mu\ f_3\rangle\,. \tag{13}$$

If $F_{++0}\,|\mu\ f_3\rangle$ is not the zero vector, then it is an eigenvector of F_{05} and F_{12} with respective eigenvalues $\mu{+}1$ and $f_3{+}1$,

$$F_{05}F_{++0}\,|\mu\ f_3\rangle = F_{++0}(F_{05}+I)|\mu\ f_3\rangle = F_{++0}(\mu{+}1)|\mu\ f_3\rangle = (\mu{+}1)F_{++0}|\mu\ f_3\rangle\,, \tag{14}$$

$$F_{12}F_{++0}\,|\mu\ f_3\rangle = F_{++0}(F_{12}+I)|\mu\ f_3\rangle = F_{++0}(f_3{+}1)|\mu\ f_3\rangle = (f_3{+}1)F_{++0}|\mu\ f_3\rangle\,. \tag{15}$$

If $F_{--0}\,|\mu\ f_3\rangle$ is not the zero vector, then it is an eigenvector of F_{05} and F_{12} with respective eigenvalues $\mu{-}1$ and $f_3{-}1$,

$$F_{05}F_{--0}\,|\mu\ f_3\rangle = F_{--0}(F_{05}-I)|\mu\ f_3\rangle = F_{--0}(\mu{-}1)|\mu\ f_3\rangle = (\mu{-}1)F_{--0}|\mu\ f_3\rangle\,, \tag{16}$$

$$F_{12}F_{--0}\,|\mu\ f_3\rangle = F_{--0}(F_{12}-I)|\mu\ f_3\rangle = F_{--0}(f_3{-}1)|\mu\ f_3\rangle = (f_3{-}1)F_{--0}|\mu\ f_3\rangle\,. \tag{17}$$

If $F_{+-0}\,|\mu\ f_3\rangle$ is not the zero vector, then it is an eigenvector of F_{05} and F_{12} with respective eigenvalues $\mu{+}1$ and $f_3{-}1$,

$$F_{05}F_{+-0}\,|\mu\ f_3\rangle = F_{+-0}(F_{05}+I)|\mu\ f_3\rangle = F_{+-0}(\mu{+}1)|\mu\ f_3\rangle = (\mu{+}1)F_{+-0}|\mu\ f_3\rangle\,, \tag{18}$$

$$F_{12}F_{+-0}\,|\mu\ f_3\rangle = F_{+-0}(F_{12}-I)|\mu\ f_3\rangle = F_{+-0}(f_3{-}1)|\mu\ f_3\rangle = (f_3{-}1)F_{+-0}|\mu\ f_3\rangle\,. \tag{19}$$

If $F_{-+0}\,|\mu\ f_3\rangle$ is not the zero vector, then it is an eigenvector of F_{05} and F_{12} with respective eigenvalues $\mu{-}1$ and $f_3{+}1$,

$$F_{05}F_{-+0}\,|\mu\ f_3\rangle = F_{-+0}(F_{05}-I)|\mu\ f_3\rangle = F_{-+0}(\mu{-}1)|\mu\ f_3\rangle = (\mu{-}1)F_{-+0}|\mu\ f_3\rangle\,, \tag{20}$$

$$F_{12}F_{-+0}\,|\mu\ f_3\rangle = F_{-+0}(F_{12}+I)|\mu\ f_3\rangle = F_{-+0}(f_3{+}1)|\mu\ f_3\rangle = (f_3{+}1)F_{-+0}|\mu\ f_3\rangle\,. \tag{21}$$

For $n \in \{0,1,2,\ldots\}$ repeated applications of the operators, if $F^n_{++0}|\mu\ f_3\rangle$ is not the zero vector, then it is an eigenvector of F_{05} and F_{12} with respective eigenvalues $\mu{+}n$ and $f_3{+}n$, if $F^n_{--0}|\mu\ f_3\rangle$ is not the zero vector, then it is an eigenvector of F_{05} and F_{12} with respective eigenvalues $\mu{-}n$ and $f_3{-}n$, if $F^n_{+-0}|\mu\ f_3\rangle$ is not the zero vector, then it is an eigenvector of F_{05} and F_{12} with respective eigenvalues $\mu{+}n$ and $f_3{-}n$, and, if $F^n_{-+0}|\mu\ f_3\rangle$ is not the zero vector, then it is an eigenvector of F_{05} and F_{12} with respective eigenvalues $\mu{-}n$ and $f_3{+}n$.

We restrict consideration to representations with a scalar product that fulfills

$$\begin{array}{lll} (F_{05}|\psi\rangle,|\phi\rangle) = (|\psi\rangle,F_{05}|\phi\rangle)\,, & (F_{12}|\psi\rangle,|\phi\rangle) = (|\psi\rangle,F_{12}|\phi\rangle)\,, & (F_{01}|\psi\rangle,|\phi\rangle) = (|\psi\rangle,F_{01}|\phi\rangle)\,, \\ (F_{02}|\psi\rangle,|\phi\rangle) = (|\psi\rangle,F_{02}|\phi\rangle)\,, & (F_{15}|\psi\rangle,|\phi\rangle) = (|\psi\rangle,F_{15}|\phi\rangle)\,, & (F_{25}|\psi\rangle,|\phi\rangle) = (|\psi\rangle,F_{25}|\phi\rangle)\,, \end{array} \quad \text{for all } |\phi\rangle \text{ and } |\psi\rangle, \tag{22}$$

and, consequently,[1]

$$\begin{aligned} (F_{--0}|\psi\rangle,|\phi\rangle) &= \big(\tfrac{1}{2}(F_{01}-F_{25}-iF_{02}-iF_{15})|\psi\rangle,|\phi\rangle\big) = \tfrac{1}{2}(F_{01}|\psi\rangle,|\phi\rangle)-\tfrac{1}{2}(F_{25}|\psi\rangle,|\phi\rangle)+\tfrac{i}{2}(F_{02}|\psi\rangle,|\phi\rangle)+\tfrac{i}{2}(F_{15}|\psi\rangle,|\phi\rangle) \\ &= \tfrac{1}{2}(|\psi\rangle,F_{01}|\phi\rangle)-\tfrac{1}{2}(|\psi\rangle,F_{25}|\phi\rangle)+\tfrac{i}{2}(|\psi\rangle,F_{02}|\phi\rangle)+\tfrac{i}{2}(|\psi\rangle,F_{15}|\phi\rangle) = \big(|\psi\rangle,\tfrac{1}{2}(F_{01}-F_{25}+iF_{02}+iF_{15})|\phi\rangle\big) \\ &= (|\psi\rangle,F_{++0}|\phi\rangle)\,, \qquad \text{for all } |\phi\rangle \text{ and } |\psi\rangle, \end{aligned} \tag{23}$$

$$\begin{aligned} (F_{-+0}|\psi\rangle,|\phi\rangle) &= \big(-\tfrac{1}{2}(F_{01}+F_{25}+iF_{02}-iF_{15})|\psi\rangle,|\phi\rangle\big) = -\tfrac{1}{2}(F_{01}|\psi\rangle,|\phi\rangle)-\tfrac{1}{2}(F_{25}|\psi\rangle,|\phi\rangle)+\tfrac{i}{2}(F_{02}|\psi\rangle,|\phi\rangle)-\tfrac{i}{2}(F_{15}|\psi\rangle,|\phi\rangle) \\ &= -\tfrac{1}{2}(|\psi\rangle,F_{01}|\phi\rangle)-\tfrac{1}{2}(|\psi\rangle,F_{25}|\phi\rangle)+\tfrac{i}{2}(|\psi\rangle,F_{02}|\phi\rangle)-\tfrac{i}{2}(|\psi\rangle,F_{15}|\phi\rangle) = \big(|\psi\rangle,-\tfrac{1}{2}(F_{01}+F_{25}-iF_{02}+iF_{15})|\phi\rangle\big) \\ &= (|\psi\rangle,F_{+-0}|\phi\rangle)\,, \qquad \text{for all } |\phi\rangle \text{ and } |\psi\rangle, \end{aligned} \tag{24}$$

where the first and fifth equalities use Eq. (4), the second equalities use anti-linearity of the left argument of the scalar product, the third equalities use Eq. (22), and the fourth equalities use linearity of the right argument of the scalar product. A representation that fulfills Eq. (22) is called *hermitian.*

For a hermitian representation, use of $|\psi\rangle = |\mu\ f_3\rangle$ and $|\phi\rangle = |\mu\ f_3\rangle$ in Eq. (22) shows that eigenvalues of F_{05} and F_{12} are real,

$$\begin{aligned} \overline{\mu}\,\langle\mu\ f_3|\mu\ f_3\rangle &= (\mu|\mu\ f_3\rangle,|\mu\ f_3\rangle) = (F_{05}|\mu\ f_3\rangle,|\mu\ f_3\rangle) \\ &= (|\mu\ f_3\rangle,F_{05}|\mu\ f_3\rangle) = (|\mu\ f_3\rangle,\mu|\mu\ f_3\rangle) = \mu\,\langle\mu\ f_3|\mu\ f_3\rangle\,, \qquad \text{or} \qquad \overline{\mu} = \mu, \end{aligned} \tag{25}$$

$$\begin{aligned} \overline{f_3}\,\langle\mu\ f_3|\mu\ f_3\rangle &= (f_3|\mu\ f_3\rangle,|\mu\ f_3\rangle) = (F_{12}|\mu\ f_3\rangle,|\mu\ f_3\rangle) \\ &= (|\mu\ f_3\rangle,F_{12}|\mu\ f_3\rangle) = (|\mu\ f_3\rangle,f_3|\mu\ f_3\rangle) = f_3\,\langle\mu\ f_3|\mu\ f_3\rangle\,, \qquad \text{or} \qquad \overline{f_3} = f_3. \end{aligned} \tag{26}$$

We further restrict consideration to representations with a lowest eigenvalue $\mu_{\min}$ of F_{05}, called *lowest-weight* hermitian representations, and representations with a highest eigenvalue $\mu_{\max}$ of F_{05}, called *highest-weight* hermitian representations.

Let $|\mu_{\min}\ f_{3\min}\rangle$ be an eigenvector of F_{05} with lowest eigenvalue $\mu_{\min}$. The vectors $F_{--0}|\mu_{\min}\ f_{3\min}\rangle$ and $F_{-+0}|\mu_{\min}\ f_{3\min}\rangle$ must then be the zero vector in order not to be eigenvectors of F_{05} with eigenvalue lower than $\mu_{\min}$.

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[1] Operators, such as F_{++0} and F_{--0}, or F_{+-0} and F_{-+0}, which fulfill Eq. (23) or (24) are called *adjoints* of each other, often denoted by $F^\dagger_{--0} = F_{++0}$, $F^\dagger_{++0} = F_{--0}$, $F^\dagger_{-+0} = F_{+-0}$, $F^\dagger_{+-0} = F_{-+0}$. Operators, such as F_{05}, F_{12}, F_{01}, F_{02}, F_{15}, and F_{25}, which fulfill Eq. (22) are *self-adjoint*, $F^\dagger_{05} = F_{05}$, $F^\dagger_{12} = F_{12}$, $F^\dagger_{01} = F_{01}$, $F^\dagger_{02} = F_{02}$, $F^\dagger_{15} = F_{15}$, and $F^\dagger_{25} = F_{25}$.

(The operator F_{12} does not necessarily have a lowest eigenvalue.) All vectors obtained by acting with operators $\mathcal{E}$ in the enveloping algebra $\mathcal{E}(\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}})$ on the vector $|\mu_{\min}\ f_{3\min}\rangle$ are eigenvectors of the operators of Eqs. (10) and (11) with respective eigenvalues $(\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)$ and $(\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)$,

$$\begin{aligned}[(F_{05}+F_{12})(F_{05}+F_{12}-2I)-4F_{++0}F_{--0}]\,\mathcal{E}|\mu_{\min}\ f_{3\min}\rangle &= \mathcal{E}[(F_{05}+F_{12})(F_{05}+F_{12}-2I)-4F_{++0}F_{--0}]|\mu_{\min}\ f_{3\min}\rangle \\ = \mathcal{E}(\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)|\mu_{\min}\ f_{3\min}\rangle &= (\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)\mathcal{E}\,|\mu_{\min}\ f_{3\min}\rangle\,,\end{aligned} \tag{27}$$

$$\begin{aligned}[(F_{05}-F_{12})(F_{05}-F_{12}-2I)-4F_{+-0}F_{-+0}]\,\mathcal{E}|\mu_{\min}\ f_{3\min}\rangle &= \mathcal{E}[(F_{05}-F_{12})(F_{05}-F_{12}-2I)-4F_{+-0}F_{-+0}]|\mu_{\min}\ f_{3\min}\rangle \\ = \mathcal{E}(\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)|\mu_{\min}\ f_{3\min}\rangle &= (\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)\mathcal{E}\,|\mu_{\min}\ f_{3\min}\rangle\,.\end{aligned} \tag{28}$$

Actions of the operators F_{++0}, F_{--0}, F_{+-0}, F_{-+0} on the vector $|\mu\ f_3\rangle$ may be expressed as

$$\begin{aligned} F_{+-0}|\mu\ f_3\rangle &= p(\mu,f_3)|\mu{+}1\ f_3{-}1\rangle\,, & F_{++0}|\mu\ f_3\rangle &= u(\mu,f_3)|\mu{+}1\ f_3{+}1\rangle\,, \\ F_{--0}|\mu\ f_3\rangle &= w(\mu,f_3)|\mu{-}1\ f_3{-}1\rangle\,, & F_{-+0}|\mu\ f_3\rangle &= d(\mu,f_3)|\mu{-}1\ f_3{+}1\rangle\,, \end{aligned} \tag{29}$$

where $|\mu{+}1\ f_3{+}1\rangle$ is an eigenvector of F_{05} and F_{12} with respective eigenvalues $\mu{+}1$ and $f_3{+}1$, $|\mu{-}1\ f_3{-}1\rangle$ is an eigenvector of F_{05} and F_{12} with respective eigenvalues $\mu-1$ and f_3-1, $|\mu{+}1\ f_3{-}1\rangle$ is an eigenvector of F_{05} and F_{12} with respective eigenvalues $\mu{+}1$ and $f_3{-}1$, $|\mu{-}1\ f_3{+}1\rangle$ is an eigenvector of F_{05} and F_{12} with respective eigenvalues $\mu{-}1$ and $f_3{+}1$, and $u(\mu,f_3)$, $w(\mu,f_3)$, $p(\mu,f_3)$, and $d(\mu,f_3)$ are functions of μ and f_3. Two conditions on the functions $u(\mu,f_3)$, $w(\mu,f_3)$, $p(\mu,f_3)$, and $d(\mu,f_3)$ are obtained from the actions of the operators of Eqs. (10) and (11) on the vector $|\mu\ f_3\rangle$ (which equals $\mathcal{E}|\mu_{\min}\ f_{3\min}\rangle$ for an appropriate operator $\mathcal{E}$ in the enveloping algebra),

$$\begin{aligned}(\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)|\mu\ f_3\rangle &= [(F_{05}+F_{12})(F_{05}+F_{12}-2I)-4F_{++0}F_{--0}]|\mu\ f_3\rangle \\ &= [(\mu+f_3)(\mu+f_3-2)-4u(\mu-1,f_3-1)\,w(\mu,f_3)]|\mu\ f_3\rangle\,,\end{aligned} \tag{30}$$

$$\begin{aligned}(\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)|\mu\ f_3\rangle &= [(F_{05}-F_{12})(F_{05}-F_{12}-2I)-4F_{+-0}F_{-+0}]|\mu\ f_3\rangle \\ &= [(\mu-f_3)(\mu-f_3-2)-4p(\mu-1,f_3+1)\,d(\mu,f_3)]|\mu\ f_3\rangle\,,\end{aligned} \tag{31}$$

or

$$u(\mu-1,f_3-1)\,w(\mu,f_3) = [(\mu+f_3)(\mu+f_3-2)-(\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)]/4, \tag{32}$$

$$p(\mu-1,f_3+1)\,d(\mu,f_3) = [(\mu-f_3)(\mu-f_3-2)-(\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)]/4. \tag{33}$$

Two more conditions on the functions $u(\mu,f_3)$, $w(\mu,f_3)$, $p(\mu,f_3)$, and $d(\mu,f_3)$ are obtained by use of $|\psi\rangle = |\mu\ f_3\rangle$ and $|\phi\rangle{=}|\mu{-}1\ f_3{-}1\rangle$ in Eq. (23) and by use of $|\psi\rangle{=}|\mu\ f_3\rangle$ and $|\phi\rangle{=}|\mu{-}1\ f_3{+}1\rangle$ in Eq. (24),

$$\begin{aligned}\overline{w(\mu,f_3)}\,\langle\mu{-}1\ f_3{-}1|\mu{-}1\ f_3{-}1\rangle &= (w(\mu,f_3)|\mu{-}1\ f_3{-}1\rangle, |\mu{-}1\ f_3{-}1\rangle) = (F_{--0}|\mu\ f_3\rangle, |\mu{-}1\ f_3{-}1\rangle) \\ = (|\mu\ f_3\rangle, F_{++0}|\mu{-}1\ f_3{-}1\rangle) &= (|\mu\ f_3\rangle, u(\mu{-}1,f_3{-}1)|\mu\ f_3\rangle) = u(\mu{-}1,f_3{-}1)\,\langle\mu\ f_3|\mu\ f_3\rangle\,,\end{aligned} \tag{34}$$

$$\begin{aligned}\overline{d(\mu,f_3)}\,\langle\mu{-}1\ f_3{+}1|\mu{-}1\ f_3{+}1\rangle &= (d(\mu,f_3)|\mu{-}1\ f_3{+}1\rangle, |\mu{-}1\ f_3{+}1\rangle) = (F_{-+0}|\mu\ f_3\rangle, |\mu{-}1\ f_3{+}1\rangle) \\ = (|\mu\ f_3\rangle, F_{+-0}|\mu{-}1\ f_3{+}1\rangle) &= (|\mu\ f_3\rangle, p(\mu{-}1,f_3{+}1)|\mu\ f_3\rangle) = p(\mu{-}1,f_3{+}1)\,\langle\mu\ f_3|\mu\ f_3\rangle\,.\end{aligned} \tag{35}$$

Elimination of $u(\mu{-}1,f_3{-}1)$ from Eqs. (32) and (34) and elimination of $p(\mu{-}1,f_3{+}1)$ from Eqs. (33) and (35) give

$$|w(\mu,f_3)|^2\,\langle\mu{-}1\ f_3{-}1|\mu{-}1\ f_3{-}1\rangle = [(\mu+f_3)(\mu+f_3-2)-(\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)]/4\ \langle\mu\ f_3|\mu\ f_3\rangle\,, \tag{36}$$

$$|d(\mu,f_3)|^2\,\langle\mu{-}1\ f_3{+}1|\mu{-}1\ f_3{+}1\rangle = [(\mu-f_3)(\mu-f_3-2)-(\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)]/4\ \langle\mu\ f_3|\mu\ f_3\rangle\,. \tag{37}$$

Since $|w(\mu,f_3)|^2$ and the squared lengths $\langle\mu\ f_3|\mu\ f_3\rangle$ and $\langle\mu{-}1\ f_3{-}1|\mu{-}1\ f_3{-}1\rangle$ of vectors $|\mu\ f_3\rangle$ and $|\mu{-}1\ f_3{-}1\rangle$ are real and non-negative, either $[(\mu+f_3)(\mu+f_3-2)-(\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)]/4$ is real and non-negative or $|\mu\ f_3\rangle$ and $F_{--0}|\mu\ f_3\rangle = w(\mu,f_3)|\mu{-}1\ f_3{-}1\rangle$ are the zero vector. Since $|d(\mu,f_3)|^2$ and the squared lengths $\langle\mu\ f_3|\mu\ f_3\rangle$ and $\langle\mu{-}1\ f_3{+}1|\mu{-}1\ f_3{+}1\rangle$ of vectors $|\mu\ f_3\rangle$ and $|\mu{-}1\ f_3{+}1\rangle$ are real and non-negative, either $[(\mu-f_3)(\mu-f_3-2)-(\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)]/4$ is real and non-negative or $|\mu\ f_3\rangle$ and $F_{-+0}|\mu\ f_3\rangle = d(\mu,f_3)|\mu{-}1\ f_3{+}1\rangle$ are the zero vector. For $w(\mu,f_3) = |w(\mu,f_3)|\,e^{i\theta_{\mu,f_3}}$ and $d(\mu,f_3) = |d(\mu,f_3)|\,e^{i\xi_{\mu,f_3}}$ with (constrained) phase factors $e^{i\theta_{\mu,f_3}}$ and $e^{i\xi_{\mu,f_3}}$,

$$w(\mu,f_3)\sqrt{\langle\mu{-}1\ f_3{-}1|\mu{-}1\ f_3{-}1\rangle} = \sqrt{(\mu+f_3)(\mu+f_3-2)-(\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)}/2\ e^{i\theta_{\mu,f_3}}\sqrt{\langle\mu\ f_3|\mu\ f_3\rangle}, \tag{38}$$

$$d(\mu,f_3)\sqrt{\langle\mu{-}1\ f_3{+}1|\mu{-}1\ f_3{+}1\rangle} = \sqrt{(\mu-f_3)(\mu-f_3-2)-(\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)}/2\ e^{i\xi_{\mu,f_3}}\sqrt{\langle\mu\ f_3|\mu\ f_3\rangle}. \tag{39}$$

Elimination of $w(\mu,f_3)$ from Eqs. (34) and (38) and elimination of $d(\mu,f_3)$ from Eqs. (35) and (39) give

$$u(\mu{-}1,f_3{-}1)\sqrt{\langle\mu\ f_3|\mu\ f_3\rangle} = \sqrt{(\mu+f_3)(\mu+f_3-2)-(\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)}/2\ e^{-i\theta_{\mu,f_3}}\sqrt{\langle\mu{-}1\ f_3{-}1|\mu{-}1\ f_3{-}1\rangle}, \tag{40}$$

$$p(\mu{-}1,f_3{+}1)\sqrt{\langle\mu\ f_3|\mu\ f_3\rangle} = \sqrt{(\mu-f_3)(\mu-f_3-2)-(\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)}/2\ e^{-i\xi_{\mu,f_3}}\sqrt{\langle\mu{-}1\ f_3{+}1|\mu{-}1\ f_3{+}1\rangle}. \tag{41}$$

Substitution of $\mu{+}1$ for μ and $f_3{+}1$ for f_3 in Eq. (40) and substitution of $\mu{+}1$ for μ and $f_3{-}1$ for f_3 in Eq. (41) give

$$u(\mu,f_3)\sqrt{\langle\mu{+}1\ f_3{+}1|\mu{+}1\ f_3{+}1\rangle} = \sqrt{(\mu+f_3+2)(\mu+f_3)-(\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)}/2\ e^{-i\theta_{\mu+1,f_3+1}}\sqrt{\langle\mu\ f_3|\mu\ f_3\rangle}, \tag{42}$$

$$p(\mu,f_3)\sqrt{\langle\mu{+}1\ f_3{-}1|\mu{+}1\ f_3{-}1\rangle} = \sqrt{(\mu-f_3+2)(\mu-f_3)-(\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)}/2\ e^{-i\xi_{\mu+1,f_3-1}}\sqrt{\langle\mu\ f_3|\mu\ f_3\rangle}. \tag{43}$$

Elimination of $u(\mu,f_3)$, $w(\mu,f_3)$, $p(\mu,f_3)$, and $d(\mu,f_3)$ from Eqs. (29), (42), (38), (43), and (39) gives

$$F_{++0}\frac{|\mu\ f_3\rangle}{\sqrt{\langle\mu\ f_3|\mu\ f_3\rangle}}=\sqrt{(\mu+f_3+2)(\mu+f_3)-(\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)}/2\ e^{-i\theta_{\mu+1,f_3+1}}\frac{|\mu+1\ f_3+1\rangle}{\sqrt{\langle\mu+1\ f_3+1|\mu+1\ f_3+1\rangle}},\quad(44)$$

$$F_{--0}\frac{|\mu\ f_3\rangle}{\sqrt{\langle\mu\ f_3|\mu\ f_3\rangle}}=\sqrt{(\mu+f_3)(\mu+f_3-2)-(\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)}/2\ e^{i\theta_{\mu,f_3}}\frac{|\mu-1\ f_3-1\rangle}{\sqrt{\langle\mu-1\ f_3-1|\mu-1\ f_3-1\rangle}},\quad(45)$$

$$F_{+-0}\frac{|\mu\ f_3\rangle}{\sqrt{\langle\mu\ f_3|\mu\ f_3\rangle}}=\sqrt{(\mu-f_3+2)(\mu-f_3)-(\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)}/2\ e^{-i\xi_{\mu+1,f_3-1}}\frac{|\mu+1\ f_3-1\rangle}{\sqrt{\langle\mu+1\ f_3-1|\mu+1\ f_3-1\rangle}},\quad(46)$$

$$F_{-+0}\frac{|\mu\ f_3\rangle}{\sqrt{\langle\mu\ f_3|\mu\ f_3\rangle}}=\sqrt{(\mu-f_3)(\mu-f_3-2)-(\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)}/2\ e^{i\xi_{\mu,f_3}}\frac{|\mu-1\ f_3+1\rangle}{\sqrt{\langle\mu-1\ f_3+1|\mu-1\ f_3+1\rangle}}.\quad(47)$$

Substitution of $\mu=\mu_{\min}+n$ and $f_3=f_{3\min}+m$ in Eqs. (12), (13), (44), (45), (46), and (47) gives

$$F_{05}|\mu_{\min}+n\ f_{3\min}+m\rangle=(\mu_{\min}+n)|\mu_{\min}+n\ f_{3\min}+m\rangle\,,\quad(48)$$

$$F_{12}|\mu_{\min}+n\ f_{3\min}+m\rangle=(f_{3\min}+m)|\mu_{\min}+n\ f_{3\min}+m\rangle\,,\quad(49)$$

$$F_{++0}\frac{|\mu_{\min}+n\ f_{3\min}+m\rangle}{\sqrt{\langle\mu_{\min}+n\ f_{3\min}+m|\mu_{\min}+n\ f_{3\min}+m\rangle}}=\sqrt{(2\mu_{\min}+2f_{3\min}+n+m)(n+m+2)}/2\ e^{-i\theta_{\mu_{\min}+n+1,f_{3\min}+m+1}}$$
$$|\mu_{\min}+n+1\ f_{3\min}+m+1\rangle/\sqrt{\langle\mu_{\min}+n+1\ f_{3\min}+m+1|\mu_{\min}+n+1\ f_{3\min}+m+1\rangle},\quad(50)$$

$$F_{--0}\frac{|\mu_{\min}+n\ f_{3\min}+m\rangle}{\sqrt{\langle\mu_{\min}+n\ f_{3\min}+m|\mu_{\min}+n\ f_{3\min}+m\rangle}}=\sqrt{(2\mu_{\min}+2f_{3\min}+n+m-2)(n+m)}/2\ e^{i\theta_{\mu_{\min}+n,f_{3\min}+m}}$$
$$|\mu_{\min}+n-1\ f_{3\min}+m-1\rangle/\sqrt{\langle\mu_{\min}+n-1\ f_{3\min}+m-1|\mu_{\min}+n-1\ f_{3\min}+m-1\rangle},\quad(51)$$

$$F_{+-0}\frac{|\mu_{\min}+n\ f_{3\min}+m\rangle}{\sqrt{\langle\mu_{\min}+n\ f_{3\min}+m|\mu_{\min}+n\ f_{3\min}+m\rangle}}=\sqrt{(2\mu_{\min}-2f_{3\min}+n-m)(n-m+2)}/2\ e^{-i\xi_{\mu_{\min}+n+1,f_{3\min}+m-1}}$$
$$|\mu_{\min}+n+1\ f_{3\min}+m-1\rangle/\sqrt{\langle\mu_{\min}+n+1\ f_{3\min}+m-1|\mu_{\min}+n+1\ f_{3\min}+m-1\rangle},\quad(52)$$

$$F_{-+0}\frac{|\mu_{\min}+n\ f_{3\min}+m\rangle}{\sqrt{\langle\mu_{\min}+n\ f_{3\min}+m|\mu_{\min}+n\ f_{3\min}+m\rangle}}=\sqrt{(2\mu_{\min}-2f_{3\min}+n-m-2)(n-m)}/2\ e^{i\xi_{\mu_{\min}+n,f_{3\min}+m}}$$
$$|\mu_{\min}+n-1\ f_{3\min}+m+1\rangle/\sqrt{\langle\mu_{\min}+n-1\ f_{3\min}+m+1|\mu_{\min}+n-1\ f_{3\min}+m+1\rangle}.\quad(53)$$

Actions of the operators F_{05}, F_{12}, F_{++0}, F_{--0}, F_{+-0}, and F_{-+0} on the unit length vectors

$$|n\ m\rangle=\frac{e^{i\alpha_{n,m}}|\mu_{\min}+n\ f_{3\min}+m\rangle}{\sqrt{\langle\mu_{\min}+n\ f_{3\min}+m\,|\mu_{\min}+n\ f_{3\min}+m\rangle}},\qquad\text{where}\quad \begin{aligned}\alpha_{n,m}&=\alpha_{n-1,m-1}-\theta_{\mu_{\min}+n,f_{3\min}+m}\\&=\alpha_{n-1,m+1}-\xi_{\mu_{\min}+n,f_{3\min}+m}\end{aligned}\quad(\text{modulo }2\pi),\quad(54)$$

are given by

$$F_{05}|n\ m\rangle=(\mu_{\min}+n)|n\ m\rangle\,,\quad(55)$$

$$F_{12}|n\ m\rangle=(f_{3\min}+m)|n\ m\rangle\,,\quad(56)$$

$$F_{++0}|n\ m\rangle=\sqrt{(2\mu_{\min}+2f_{3\min}+n+m)(n+m+2)}/2\,|n+1\ m+1\rangle\,,\quad(57)$$

$$F_{--0}|n\ m\rangle=\sqrt{(2\mu_{\min}+2f_{3\min}+n+m-2)(n+m)}/2\,|n-1\ m-1\rangle\,,\quad(58)$$

$$F_{+-0}|n\ m\rangle=\sqrt{(2\mu_{\min}-2f_{3\min}+n-m)(n-m+2)}/2\,|n+1\ m-1\rangle\,,\quad(59)$$

$$F_{-+0}|n\ m\rangle=\sqrt{(2\mu_{\min}-2f_{3\min}+n-m-2)(n-m)}/2\,|n-1\ m+1\rangle\quad(60)$$

and actions of the operators of Eqs. (7), (8), (10), and (11) on the unit length vectors $|n\ m\rangle$ are given by

$$(F_{05}F_{05}+F_{12}F_{12}-F_{01}F_{01}-F_{02}F_{02}-F_{15}F_{15}-F_{25}F_{25})|n\ m\rangle=\left[\mu_{\min}(\mu_{\min}-2)+f_{3\min}^2\right]|n\ m\rangle\,,\quad(61)$$

$$(F_{05}F_{12}-F_{02}F_{15}+F_{01}F_{25})|n\ m\rangle=(\mu_{\min}-1)f_{3\min}\,|n\ m\rangle\,,\quad(62)$$

$$[(F_{05}+F_{12})(F_{05}+F_{12}-2I)-4F_{++0}F_{--0}]|n\ m\rangle=(\mu_{\min}+f_{3\min})(\mu_{\min}+f_{3\min}-2)|n\ m\rangle\,,\quad(63)$$

$$[(F_{05}-F_{12})(F_{05}-F_{12}-2I)-4F_{+-0}F_{-+0}]|n\ m\rangle=(\mu_{\min}-f_{3\min})(\mu_{\min}-f_{3\min}-2)|n\ m\rangle\,.\quad(64)$$

Use of Eqs. (22), (25), and (26) shows that, if $n'\neq n$ or if $m'\neq m$, then the unit length vectors $|n'\ m'\rangle$ and $|n\ m\rangle$ are orthogonal,

$$\begin{aligned}(\mu_{\min}+n')\,\langle n'\ m'|n\ m\rangle&=\overline{(\mu_{\min}+n')}\,\langle n'\ m'|n\ m\rangle=((\mu_{\min}+n')|n'\ m'\rangle,|n\ m\rangle)=(F_{05}|n'\ m'\rangle,|n\ m\rangle)\\&=(|n'\ m'\rangle,F_{05}|n\ m\rangle)=(|n'\ m'\rangle,(\mu_{\min}+n)|n\ m\rangle)=(\mu_{\min}+n)\,\langle n'\ m'|n\ m\rangle\,,\end{aligned}\quad(65)$$

or

$$(n'-n)\,\langle n'\ m'|n\ m\rangle=0,\qquad\text{or, if }\ n'\neq n,\ \text{ then}\qquad\langle n'\ m'|n\ m\rangle=0,\quad(66)$$

and

$$\begin{aligned}(f_{3\min}+m')\,\langle n'\ m'|n\ m\rangle &= \overline{(f_{3\min}+m')}\,\langle n'\ m'|n\ m\rangle = ((f_{3\min}+m')|n'\ m'\rangle, |n\ m\rangle) = (F_{12}|n'\ m'\rangle, |n\ m\rangle) \\ &= (|n'\ m'\rangle, F_{12}|n\ m\rangle) = (|n'\ m'\rangle, (f_{3\min}+m)|n\ m\rangle) = (f_{3\min}+m)\,\langle n'\ m'|n\ m\rangle\,,\end{aligned} \tag{67}$$

or

$$(m'-m)\,\langle n'\ m'|n\ m\rangle = 0, \qquad \text{or, if } m' \neq m, \text{ then} \qquad \langle n'\ m'|n\ m\rangle = 0. \tag{68}$$

For all lowest-weight hermitian irreducible ray representations (lowest-weight *hermirreps*) $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}, f_{3\min})$ of the Lie algebra of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$ the lowest value of n is 0 and arguments within square roots on right hand sides of Eqs. (57), (58), (59), and (60) are never negative,

$$\begin{aligned}&0 \leq (2\mu_{\min}+2f_{3\min}+n+m)(n+m+2)\,, \\ &0 \leq (2\mu_{\min}+2f_{3\min}+n+m-2)(n+m)\,, \\ &0 \leq (2\mu_{\min}-2f_{3\min}+n-m)(n-m+2)\,, \\ &0 \leq (2\mu_{\min}-2f_{3\min}+n-m-2)(n-m)\,.\end{aligned} \tag{69}$$

For $n=0$ and $m=0$, Eq. (69) reduces to

$$|f_{3\min}| \leq \mu_{\min}. \tag{70}$$

If $|f_{3\min}|=\mu_{\min}$ and $f_{3\min}=0$, then actions of all basis operators of Eq. (4) on the unit length vector $|n{=}0\ m{=}0\rangle$ give the zero vector, no operator transforms away from $|n{=}0\ m{=}0\rangle$, and $|n{=}0\ m{=}0\rangle$ spans the 1-dimensional vector space $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}{=}0, f_{3\min}{=}0)$ of the lowest-weight hermirrep $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}{=}0, f_{3\min}{=}0)$ of the Lie algebra of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$.

If $|f_{3\min}|=\mu_{\min}$ and $0<f_{3\min}$, then Eq. (69) reduces to

$$\begin{aligned}&0 \leq (4f_{3\min}+n+m)(n+m+2)\,, \\ &0 \leq (4f_{3\min}+n+m-2)(n+m)\,, \\ &0 \leq (n-m)(n-m+2)\,, \\ &0 \leq (n-m-2)(n-m)\,,\end{aligned} \tag{71}$$

the operators F_{++0} and F_{--0} connect vectors $|n{=}0\ m{=}0\rangle$, $|n{=}1\ m{=}1\rangle$, $|n{=}2\ m{=}2\rangle$, ..., on which actions of the operators F_{+-0} and F_{-+0} always give the zero vector, and the vectors $|n{=}0\ m{=}0\rangle$, $|n{=}1\ m{=}1\rangle$, $|n{=}2\ m{=}2\rangle$, ..., span the ∞-dimensional vector space $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|{=}\mu_{\min}, 0{<}f_{3\min})$ of the lowest-weight hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|{=}\mu_{\min}, 0{<}f_{3\min})$ of the Lie algebra of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$.

If $|f_{3\min}|=\mu_{\min}$ and $f_{3\min}<0$, then Eq. (69) reduces to

$$\begin{aligned}&0 \leq (n+m)(n+m+2)\,, \\ &0 \leq (n+m-2)(n+m)\,, \\ &0 \leq (-4f_{3\min}+n-m)(n-m+2)\,, \\ &0 \leq (-4f_{3\min}+n-m-2)(n-m)\,,\end{aligned} \tag{72}$$

the operators F_{+-0} and F_{-+0} connect vectors $|n{=}0\ m{=}0\rangle$, $|n{=}1\ m{=}{-1}\rangle$, $|n{=}2\ m{=}{-2}\rangle$, ..., on which actions of the operators F_{++0} and F_{--0} always give the zero vector, and the vectors $|n{=}0\ m{=}0\rangle$, $|n{=}1\ m{=}{-1}\rangle$, $|n{=}2\ m{=}{-2}\rangle$, ..., span the ∞-dimensional vector space $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|{=}\mu_{\min}, f_{3\min}{<}0)$ of the lowest-weight hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|{=}\mu_{\min}, f_{3\min}{<}0)$ of the Lie algebra of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$.

If $|f_{3\min}|<\mu_{\min}$ and $f_{3\min}\in\Re$, then the operators F_{++0}, F_{--0}, F_{+-0}, and F_{-+0} connect vectors $|n\ m\rangle$ given by the dot pattern shown in Figure 1 and the vectors span the ∞-dimensional vector space $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|{<}\mu_{\min}, f_{3\min}{\in}\Re)$ of the lowest-weight hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|{<}\mu_{\min}, f_{3\min}{\in}\Re)$ of the Lie algebra of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$.

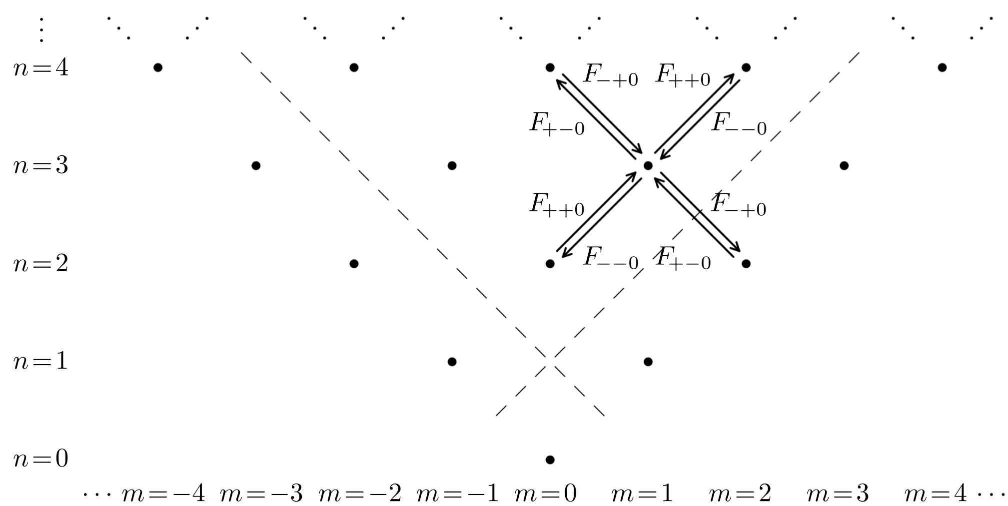

Figure 1 Dot pattern of subspaces of $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|{<}\mu_{\min}, f_{3\min}{\in}\Re)$ spanned by basis vectors $|n\ m\rangle$. Arrows show how the operators F_{++0}, F_{--0}, F_{+-0}, and F_{-+0} transform to and from the subspace spanned by $|n{=}3\ m{=}1\rangle$.

A lowest-weight hermirrep $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\rm min}, f_{3{\rm min}})$ of the Lie algebra of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$ is characterized by the hermirrep $\mathcal{D}_{F_{05},F_{12}}(\mu_{\rm min}, f_{3{\rm min}})$ of the subalgebra $\mathrm{SO}(2)_{F_{05}} \times \mathrm{SO}(2)_{F_{12}}$[2] in the eigenspace of F_{05} with lowest eigenvalue $\mu_{\rm min}$ of F_{05} and exists if and only if

$$|f_{3{\rm min}}| \leq \mu_{\rm min} < \infty \qquad \text{and} \qquad f_{3{\rm min}} \in \Re. \tag{73}$$

The vector space $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\rm min}, f_{3{\rm min}})$ of a hermirrep $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\rm min}, f_{3{\rm min}})$ is spanned by a basis of eigenvectors $|n\ m\rangle$ of F_{05} and F_{12} for which

$$\begin{array}{llll}
(n{+}m)/2 = 0 & \text{and } (n{-}m)/2 = 0 & \text{if} \quad |f_{3{\rm min}}| = \mu_{\rm min} & \text{and } f_{3{\rm min}} = 0,\\
(n{+}m)/2 \in \{0,1,2,\cdots\} & \text{and } (n{-}m)/2 = 0 & \text{if} \quad |f_{3{\rm min}}| = \mu_{\rm min} & \text{and } f_{3{\rm min}} > 0,\\
(n{+}m)/2 = 0 & \text{and } (n{-}m)/2 \in \{0,1,2,\cdots\} & \text{if} \quad |f_{3{\rm min}}| = \mu_{\rm min} & \text{and } f_{3{\rm min}} < 0,\\
(n{+}m)/2 \in \{0,1,2,\cdots\} & \text{and } (n{-}m)/2 \in \{0,1,2,\cdots\} & \text{if} \quad |f_{3{\rm min}}| < \mu_{\rm min} < \infty & \text{and } f_{3{\rm min}} \in \Re,
\end{array} \tag{74}$$

and on which the basis operators of Eq. (4) have actions given in Eqs. (55), (56), (57, (58), (59), and (60) and the operators of Eqs. (7), (8), (10), and (11) have actions given in Eqs. (61), (62), (63), and (64). The vector space $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\rm min}, f_{3{\rm min}})$ has dimension 1 if $\mu_{\rm min} = 0$ and dimension ∞ if $0 < \mu_{\rm min} < \infty$. The basis vectors $|n\ m\rangle$ are unit length and orthogonal,

$$\langle n'\ m'|n\ m\rangle = \delta_{n'n}\delta_{m'm} = \begin{cases} 1, & \text{if } n' = n \text{ and } m' = m,\\ 0, & \text{if } n' \neq n \text{ or } m' \neq m. \end{cases} \tag{75}$$

A highest-weight hermirrep $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\rm max}, f_{3{\rm min}})$ of the Lie algebra of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$ is characterized by the hermirrep $\mathcal{D}_{F_{05},F_{12}}(\mu_{\rm max}, f_{3{\rm min}})$ of the subalgebra $\mathrm{SO}(2)_{F_{05}} \times \mathrm{SO}(2)_{F_{12}}$[2] in the eigenspace of F_{05} with highest eigenvalue $\mu_{\rm max}$ of F_{05} and exists if and only if

$$-\infty < \mu_{\rm max} \leq -|f_{3{\rm min}}| \qquad \text{and} \qquad f_{3{\rm min}} \in \Re. \tag{76}$$

The vector space $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\rm max}, f_{3{\rm min}})$ of a hermirrep $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\rm max}, f_{3{\rm min}})$ is spanned by a basis of eigenvectors $|n\ m\rangle$ of F_{05} and F_{12} for which

$$\begin{array}{llll}
(n{+}m)/2 = 0 & \text{and } (n{-}m)/2 = 0 & \text{if} \quad \mu_{\rm max} = -|f_{3{\rm max}}| & \text{and } f_{3{\rm max}} = 0,\\
(n{+}m)/2 = 0 & \text{and } (n{-}m)/2 \in \{0,-1,-2,\cdots\} & \text{if} \quad \mu_{\rm max} = -|f_{3{\rm max}}| & \text{and } f_{3{\rm max}} > 0,\\
(n{+}m)/2 \in \{0,-1,-2,\cdots\} & \text{and } (n{-}m)/2 = 0 & \text{if} \quad \mu_{\rm max} = -|f_{3{\rm max}}| & \text{and } f_{3{\rm max}} < 0,\\
(n{+}m)/2 \in \{0,-1,-2,\cdots\} & \text{and } (n{-}m)/2 \in \{0,-1,-2,\cdots\} & \text{if} \quad -\infty < \mu_{\rm max} < -|f_{3{\rm max}}| & \text{and } f_{3{\rm max}} \in \Re,
\end{array} \tag{77}$$

and on which the basis operators of Eq. (4) have actions

$$F_{05}|n\ m\rangle = (\mu_{\rm max}{+}n)|n\ m\rangle\,, \tag{78}$$

$$F_{12}|n\ m\rangle = (f_{3{\rm max}}{+}m)|n\ m\rangle\,, \tag{79}$$

$$F_{++0}|n\ m\rangle = \sqrt{(2\mu_{\rm max}{+}2f_{3{\rm max}}{+}n{+}m{+}2)(n{+}m)/2}\,|n{+}1\ m{+}1\rangle\,, \tag{80}$$

$$F_{--0}|n\ m\rangle = \sqrt{(2\mu_{\rm max}{+}2f_{3{\rm max}}{+}n{+}m)(n{+}m{-}2)/2}\,|n{-}1\ m{-}1\rangle\,, \tag{81}$$

$$F_{+-0}|n\ m\rangle = \sqrt{(2\mu_{\rm max}{-}2f_{3{\rm max}}{+}n{-}m{+}2)(n{-}m)/2}\,|n{+}1\ m{-}1\rangle\,, \tag{82}$$

$$F_{-+0}|n\ m\rangle = \sqrt{(2\mu_{\rm max}{-}2f_{3{\rm max}}{+}n{-}m)(n{-}m{-}2)/2}\,|n{-}1\ m{+}1\rangle \tag{83}$$

and actions of the operators of Eqs. (7), (8), (10), and (11) on the unit length vectors $|n\ m\rangle$ are given by

$$(F_{05}F_{05}{+}F_{12}F_{12}{-}F_{01}F_{01}{-}F_{02}F_{02}{-}F_{15}F_{15}{-}F_{25}F_{25})|n\ m\rangle = \left[\mu_{\rm max}(\mu_{\rm max}{+}2){+}f_{3{\rm max}}^2\right]|n\ m\rangle\,, \tag{84}$$

$$(F_{05}F_{12}{-}F_{02}F_{15}{+}F_{01}F_{25})|n\ m\rangle = (\mu_{\rm max}{+}1)f_{3{\rm max}}\,|n\ m\rangle\,, \tag{85}$$

$$[(F_{05}{+}F_{12})(F_{05}{+}F_{12}{+}2I){-}4F_{--0}F_{++0}]|n\ m\rangle = (\mu_{\rm max}{+}f_{3{\rm max}})(\mu_{\rm max}{+}f_{3{\rm max}}{+}2)|n\ m\rangle\,, \tag{86}$$

$$[(F_{05}{-}F_{12})(F_{05}{-}F_{12}{+}2I){-}4F_{-+0}F_{+-0}]|n\ m\rangle = (\mu_{\rm max}{-}f_{3{\rm max}})(\mu_{\rm max}{-}f_{3{\rm max}}{+}2)|n\ m\rangle\,. \tag{87}$$

The vector space $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\rm max}, f_{3{\rm max}})$ has dimension 1 if $\mu_{\rm max} = 0$ and dimension ∞ if $-\infty < \mu_{\rm max} < 0$. The basis vectors $|n\ m\rangle$ are unit length and orthogonal,

$$\langle n'\ m'|n\ m\rangle = \delta_{n'n}\delta_{m'm} = \begin{cases} 1, & \text{if } n' = n \text{ and } m' = m,\\ 0, & \text{if } n' \neq n \text{ or } m' \neq m. \end{cases} \tag{88}$$

If $-\infty < \mu_{\rm max} < -|f_{3{\rm max}}|$ and $f_{3{\rm max}} \in \Re$, then the operators F_{++0}, F_{--0}, F_{+-0}, and F_{-+0} connect vectors $|n\ m\rangle$ given by the dot pattern shown in Figure 2.

Figure 3 shows all combinations of values of $\mu_{\rm min}$ and $f_{3{\rm min}}$ in Eq. (74) for which lowest-weight hermirreps of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$ exist and all combinations of values of $\mu_{\rm max}$ and $f_{3{\rm max}}$ in Eq. (77) for which highest-weight hermirreps of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$ exist.

[2] Hermirreps $\mathcal{D}_{F_{05},F_{12}}(\mu, f_3)$ of $\mathrm{SO}(2)_{F_{05}} \times \mathrm{SO}(2)_{F_{12}}$ are characterized by the eigenvalue μ of F_{05} and the eigenvalue f_3 of F_{12} and exist if and only if $\mu \in \Re$ and $f_3 \in \Re$; the vector space $\mathcal{H}_{F_{05},F_{12}}(\mu, f_3)$ of a hermirrep $\mathcal{D}_{F_{05},F_{12}}(\mu, f_3)$ of $\mathrm{SO}(2)_{F_{05}} \times \mathrm{SO}(2)_{F_{12}}$ has dimension 1.

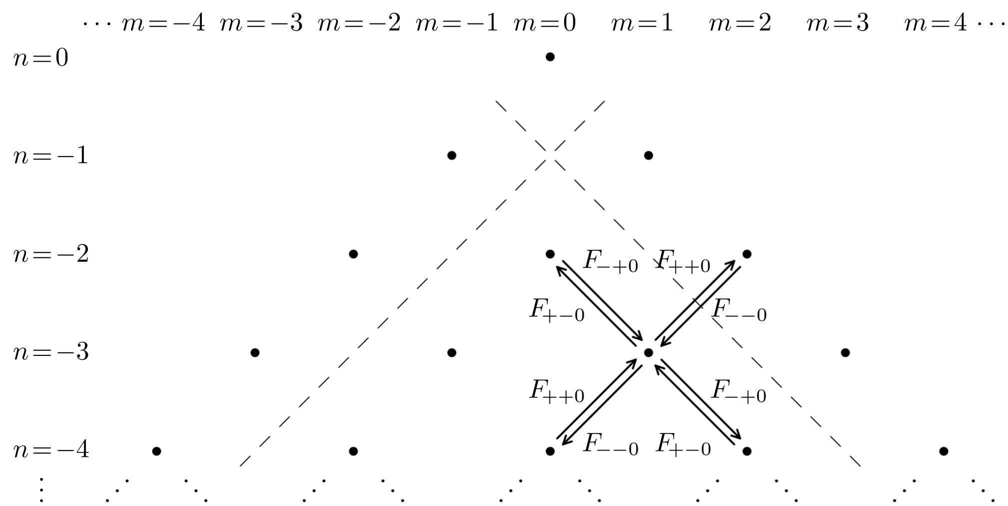

Figure 2 Dot pattern of subspaces of $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}<-|f_{3\max}|\,, f_{3\max}\in\Re)$ spanned by basis vectors $|n\ m\rangle$. Arrows show how the operators F_{++0}, F_{--0}, F_{+-0}, F_{-+0} transform to and from the subspace spanned by $|n=-3\ m=1\rangle$.

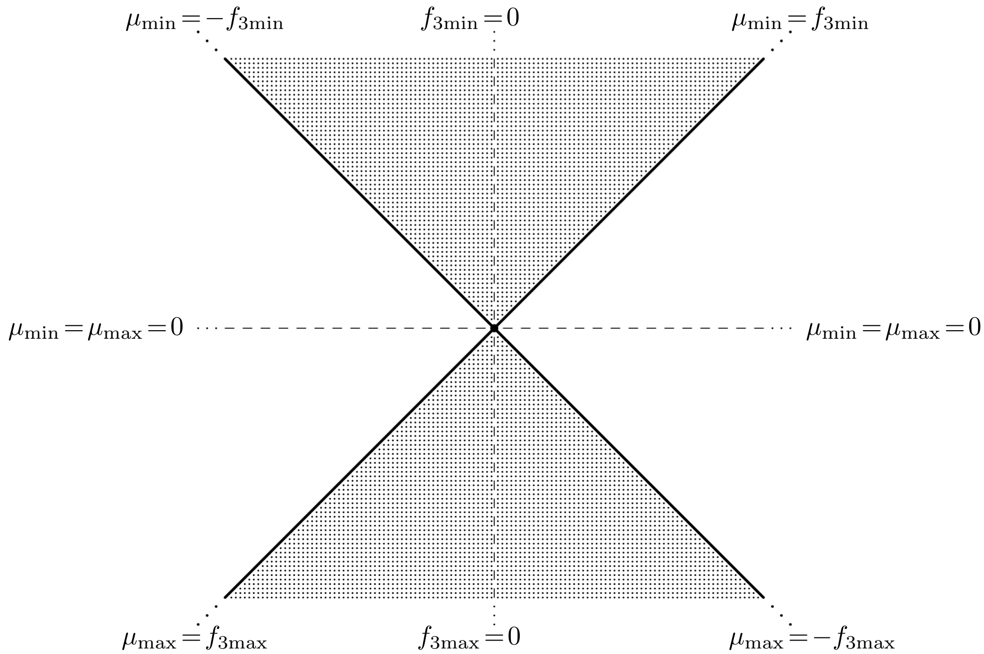

Figure 3 The diagonal lines and shaded area show all values of $(\mu_{\min}, f_{3\min})$ in Eq. (74) and all values of $(\mu_{\max}, f_{3\max})$ in Eq. (77) for which lowest-weight hermirreps and highest-weight hermirreps of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$ exist.

A lowest-weight hermirrep space $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty, f_{3\min}\in\Re)$ of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$ reduces into an infinite direct sum of lowest-weight hermirrep spaces $\mathcal{H}_{F_{05},F_{01},F_{15}}(\mu)$ of the $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ subalgebra,[3]

$$\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty, f_{3\min}\in\Re) = \sum_{n=0,1,2,\cdots}^{\infty} \oplus\mathcal{H}_{F_{05},F_{01},F_{15}}(\mu_{\min}+n)\,, \tag{89}$$

A highest-weight hermirrep space $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(-\infty<\mu_{\max}<-|f_{3\max}|\,, f_{3\min}\in\Re)$ of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$ reduces into an infinite direct sum of highest-weight hermirrep spaces $\mathcal{H}_{F_{05},F_{01},F_{15}}(\mu)$ of the $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ subalgebra,[3]

$$\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(-\infty<\mu_{\max}<-|f_{3\max}|\,, f_{3\max}\in\Re) = \sum_{n=0,-1,-2,\cdots}^{-\infty} \oplus\mathcal{H}_{F_{05},F_{01},F_{15}}(\mu_{\max}+n)\,. \tag{90}$$

The hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(0,0)$, $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|=\mu_{\min}, 0<f_{3\min})$, $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|=\mu_{\min}, f_{3\min}<0)$, $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}=-|f_{3\max}|\,, 0<f_{3\max})$, $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}=-|f_{3\max}|\,, f_{3\max}<0)$ of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$ remain irreducible with respect to $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$.

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[3] Lowest-weight hermirreps $\mathcal{D}_{F_{05},F_{01},F_{15}}(0\le\mu_{1\min})$ of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ are characterized by the lowest eigenvalue $\mu_{1\min}$ of F_{05}. Highest-weight hermirreps $\mathcal{D}_{F_{05},F_{01},F_{15}}(\mu_{1\max}\le 0)$ of $\mathrm{SO}(2,1)_{F_{05},F_{01},F_{15}}$ are characterized by the highest eigenvalue $\mu_{1\max}$ of F_{05}. $\mathcal{H}_{F_{05},F_{01},F_{15}}(0<\mu_{1\min})$ and $\mathcal{H}_{F_{05},F_{01},F_{15}}(\mu_{1\max}<0)$ are ∞-dimensional; $\mathcal{H}_{F_{05},F_{01},F_{15}}(0)$ is 1-dimensional.

In the limit $\mu_{\min}=f_{3\min}\to 0$ through the continuous sequence of hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}=f_{3\min},0<f_{3\min})$, those matrix elements of F_{++0} and F_{--0} that connect the subspace of $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}=f_{3\min},0<f_{3\min})$ for which $n=0$ to the remaining subspace become zero and the hermirrep splits into the direct sum of two hermirreps (separated by the part of the dashed line of slope -1 below the dashed line of slope 1 in Figure 1),

$$\begin{aligned}&\lim_{\mu_{\min}=f_{3\min}\to 0}\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}=f_{3\min},0<f_{3\min})\\&\quad=\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}=0,f_{3\min}=0)+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}+1=1,f_{3\min}+1=1)\,.\end{aligned}\tag{91}$$

In the limit $\mu_{\min}=-f_{3\min}\to 0$ through the continuous sequence of hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}=-f_{3\min},f_{3\min}<0)$, those matrix elements of F_{+-0} and F_{-+0} that connect the subspace of $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}=-f_{3\min},f_{3\min}<0)$ for which $n=0$ to the remaining subspace become zero and the hermirrep splits into the direct sum of two hermirreps (separated by the part of the dashed line of slope 1 below the dashed line of slope -1 in Figure 1),

$$\begin{aligned}&\lim_{\mu_{\min}=-f_{3\min}\to 0}\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}=-f_{3\min},f_{3\min}<0)\\&\quad=\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}=0,f_{3\min}=0)+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}+1=1,f_{3\min}-1=-1)\,.\end{aligned}\tag{92}$$

In the limit $\mu_{\min}\to f_{3\min}>0$ through the continuous sequence of hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty,f_{3\min})$, those matrix elements of F_{+-0} and F_{-+0} that connect the subspace of $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty,f_{3\min})$ for which $n=m$ to the remaining subspace become zero and the hermirrep splits into the direct sum of two hermirreps (separated by the dashed line of slope 1 in Figure 1),

$$\begin{aligned}&\lim_{\mu_{\min}\to f_{3\min}>0}\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty,f_{3\min})\\&\quad=\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}=f_{3\min},0<f_{3\min})+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}+1=f_{3\min}+1,f_{3\min}-1>-1)\,.\end{aligned}\tag{93}$$

In the limit $\mu_{\min}\to -f_{3\min}>0$ through the continuous sequence of hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty,f_{3\min})$, those matrix elements of F_{++0} and F_{--0} that connect the subspace of $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty,f_{3\min})$ for which $n=-m$ to the remaining subspace become zero and the hermirrep splits into the direct sum of two hermirreps (separated by the dashed line of slope -1 in Figure 1),

$$\begin{aligned}&\lim_{\mu_{\min}\to -f_{3\min}>0}\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty,f_{3\min})\\&\quad=\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}=-f_{3\min},f_{3\min}<0)+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}+1=-f_{3\min}+1,f_{3\min}+1<1)\,.\end{aligned}\tag{94}$$

In the limits $\mu_{\min}\to 0$ and $f_{3\min}\to 0$ through the continuous sequence of hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty,f_{3\min})$, those matrix elements of F_{+-0} and F_{-+0} that connect subspaces of $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty,f_{3\min})$ for which $n=m$ to remaining subspaces become zero, those matrix elements of F_{++0} and F_{--0} that connect subspaces of $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty,f_{3\min})$ for which $n=-m$ to remaining subspaces become zero, and the hermirrep splits into the direct sum of four hermirreps (separated by the dashed lines of slope 1 and -1 in Figure 1),

$$\begin{aligned}&\lim_{\mu_{\min}\to 0,f_{3\min}\to 0}\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty,f_{3\min})\\&\quad=\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}=0,f_{3\min}=0)+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}+1=1,f_{3\min}+1=1)\\&\quad\quad+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}+1=1,f_{3\min}-1=-1)+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\min}+2=2,f_{3\min}=0)\,.\end{aligned}\tag{95}$$

Analogous limits through continuous sequences of highest-weight hermirreps give

$$\begin{aligned}&\lim_{\mu_{\max}=f_{3\max}\to 0}\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}=f_{3\max},f_{3\max}<0)\\&\quad=\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}=0,f_{3\max}=0)+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}-1=-1,f_{3\max}-1=-1)\,,\end{aligned}\tag{96}$$

$$\begin{aligned}&\lim_{\mu_{\max}=-f_{3\max}\to 0}\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}=-f_{3\max},0<f_{3\max})\\&\quad=\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}=0,f_{3\max}=0)+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}-1=-1,f_{3\max}+1=1)\,,\end{aligned}\tag{97}$$

$$\begin{aligned}&\lim_{\mu_{\max}\to f_{3\max}<0}\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(-\infty<\mu_{\max}<-|f_{3\max}|\,,f_{3\max})\\&\quad=\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}=f_{3\max},f_{3\max}<0)+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}-1=f_{3\max}-1,f_{3\max}+1<1)\,,\end{aligned}\tag{98}$$

$$\begin{aligned}&\lim_{\mu_{\max}\to -f_{3\max}<0}\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(-\infty<\mu_{\max}<-|f_{3\max}|\,,f_{3\max})\\&=\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}=-f_{3\max},0<f_{3\max})+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}-1=-f_{3\max}-1,f_{3\max}-1>-1)\,,\end{aligned}\tag{99}$$

$$\begin{aligned}&\lim_{\mu_{\max}\to 0,f_{3\max}\to 0}\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(-\infty<\mu_{\max}<-|f_{3\max}|\,,f_{3\max})\\&\quad=\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}=0,f_{3\max}=0)+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}-1=-1,f_{3\max}-1=-1)\\&\quad\quad+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}-1=-1,f_{3\max}+1=1)+\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(\mu_{\max}-2=-2,f_{3\max}=0)\,.\end{aligned}\tag{100}$$

These are examples of *representation splitting*.

In the limit $\mu_{\min}\to\infty$ through the continuous sequence of hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty,f_{3\min})$, the operators

$$\begin{matrix}a_{+-0}=\lim_{\mu_{\min}\to\infty}F_{+-0}/\sqrt{\mu_{\min}}, & & & a_{++0}=\lim_{\mu_{\min}\to\infty}F_{++0}/\sqrt{\mu_{\min}},\\ & I=\lim_{\mu_{\min}\to\infty}F_{05}/\mu_{\min}, & F_{12}, & \\ a_{--0}=\lim_{\mu_{\min}\to\infty}F_{--0}/\sqrt{\mu_{\min}}, & & & a_{-+0}=\lim_{\mu_{\min}\to\infty}F_{-+0}/\sqrt{\mu_{\min}}\end{matrix}\tag{101}$$

have the commutation relations

$$\begin{gathered}
[I,I]=0,\quad [F_{12},F_{12}]=0,\quad [a_{++0},a_{++0}]=0,\quad [a_{--0},a_{--0}]=0,\quad [a_{+-0},a_{+-0}]=0,\quad [a_{-+0},a_{-+0}]=0,\\
[I,F_{12}]=0,\quad [a_{++0},a_{--0}]=-I,\quad [a_{+-0},a_{-+0}]=-I,\\
[F_{12},I]=0,\quad [a_{--0},a_{++0}]=I,\quad [a_{-+0},a_{+-0}]=I,\\
[a_{++0},a_{+-0}]=0,\quad [a_{++0},a_{-+0}]=0,\quad [a_{--0},a_{+-0}]=0,\quad [a_{--0},a_{-+0}]=0,\\
[a_{+-0},a_{++0}]=0,\quad [a_{-+0},a_{++0}]=0,\quad [a_{+-0},a_{--0}]=0,\quad [a_{-+0},a_{--0}]=0,\\
[I,a_{++0}]=0,\quad [I,a_{--0}]=0,\quad [I,a_{+-0}]=0,\quad [I,a_{-+0}]=0,\\
[a_{++0},I]=0,\quad [a_{--0},I]=0,\quad [a_{+-0},I]=0,\quad [a_{-+0},I]=0,\\
[F_{12},a_{++0}]=a_{++0},\quad [F_{12},a_{--0}]=-a_{--0},\quad [F_{12},a_{+-0}]=-a_{+-0},\quad [F_{12},a_{-+0}]=a_{-+0},\\
[a_{++0},F_{12}]=-a_{++0},\quad [a_{--0},F_{12}]=a_{--0},\quad [a_{+-0},F_{12}]=a_{+-0},\quad [a_{-+0},F_{12}]=-a_{-+0}
\end{gathered}\tag{102}$$

of the (semi-direct sum) Lie algebra of $\mathrm{SO}(2)_{F_{12}}\ltimes\mathrm{HW}(2)_{I,a_{++0},a_{--0},a_{+-0},a_{-+0}}$ and have the actions

$$I\,|n\ m\rangle = |n\ m\rangle\,,\tag{103}$$

$$F_{12}|n\ m\rangle = (f_{3\min}+m)|n\ m\rangle\,,\tag{104}$$

$$a_{++0}|n\ m\rangle = \sqrt{(n+m+2)/2}\,|n+1\ m+1\rangle\,,\tag{105}$$

$$a_{--0}|n\ m\rangle = \sqrt{(n+m)/2}\,|n-1\ m-1\rangle\,,\tag{106}$$

$$a_{+-0}|n\ m\rangle = \sqrt{(n-m+2)/2}\,|n+1\ m-1\rangle\,,\tag{107}$$

$$a_{-+0}|n\ m\rangle = \sqrt{(n-m)/2}\,|n-1\ m+1\rangle\,.\tag{108}$$

In the limit $\mu_{\max}\to-\infty$ through the continuous sequence of hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(-\infty<\mu_{\max}<-|f_{3\max}|\,,$ $f_{3\max})$, the operators

$$\begin{matrix}
a_{--0}=\lim_{\mu_{\max}\to\infty}F_{+-0}/\sqrt{-\mu_{\max}}, & & a_{-+0}=\lim_{\mu_{\max}\to\infty}F_{++0}/\sqrt{-\mu_{\max}},\\
 & I=\lim_{\mu_{\max}\to\infty}F_{05}/\mu_{\max},\qquad F_{12}, & \\
a_{+-0}=\lim_{\mu_{\max}\to\infty}F_{--0}/\sqrt{-\mu_{\max}}, & & a_{++0}=\lim_{\mu_{\max}\to\infty}F_{-+0}/\sqrt{-\mu_{\max}}
\end{matrix}\tag{109}$$

also have the commutation relations of Eq. (102) and the actions of Eqs. (103), (104), (105), (106), (107), and (108),

$$I\,|n\ m\rangle = |n\ m\rangle\,,\tag{110}$$

$$F_{12}|n\ m\rangle = (f_{3\max}+m)|n\ m\rangle\,,\tag{111}$$

$$a_{++0}|n\ m\rangle = \sqrt{-(n-m-2)/2}\,|n-1\ m+1\rangle\,,\tag{112}$$

$$a_{--0}|n\ m\rangle = \sqrt{-(n-m)/2}\,|n+1\ m-1\rangle\,,\tag{113}$$

$$a_{+-0}|n\ m\rangle = \sqrt{-(n+m-2)/2}\,|n-1\ m-1\rangle\,,\tag{114}$$

$$a_{-+0}|n\ m\rangle = \sqrt{-(n+m)/2}\,|n+1\ m+1\rangle\,;\tag{115}$$

that these are the same actions as in Eqs. (103), (104), (105), (106), (107), and (108) follows from $f_{3\min}=f_{3\max}$, the change of variable $n=-k$, with $n\in\{0,-1,-2,\ldots\}$ and $k\in\{0,1,2,\ldots\}$,

$$I\,|-k\ m\rangle = |-k\ m\rangle\,,\tag{116}$$

$$F_{12}|-k\ m\rangle = (f_{3\max}+m)|-k\ m\rangle\,,\tag{117}$$

$$a_{++0}|-k\ m\rangle = \sqrt{(k+m+2)/2}\,|-k-1\ m+1\rangle\,,\tag{118}$$

$$a_{--0}|-k\ m\rangle = \sqrt{(k+m)/2}\,|-k+1\ m-1\rangle\,,\tag{119}$$

$$a_{+-0}|-k\ m\rangle = \sqrt{(k-m+2)/2}\,|-k-1\ m-1\rangle\,,\tag{120}$$

$$a_{-+0}|-k\ m\rangle = \sqrt{(k-m)/2}\,|-k+1\ m+1\rangle\,,\tag{121}$$

and use of $|-\arg\ m\rangle = |\arg\ m\}$ to denote sign changes of the entire first arguments of the basis vectors:

$$I\,|k\ m\} = |k\ m\}\,,\tag{122}$$

$$F_{12}|k\ m\} = (f_{3\max}+m)|k\ m\}\,,\tag{123}$$

$$a_{++0}|k\ m\} = \sqrt{(k+m+2)/2}\,|k+1\ m+1\}\,,\tag{124}$$

$$a_{--0}|k\ m\} = \sqrt{(k+m)/2}\,|k-1\ m-1\}\,,\tag{125}$$

$$a_{+-0}|k\ m\} = \sqrt{(k-m+2)/2}\,|k+1\ m-1\}\,,\tag{126}$$

$$a_{-+0}|k\ m\} = \sqrt{(k-m)/2}\,|k-1\ m+1\}\,.\tag{127}$$

These are examples of *contraction* through continuous sequences of hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|<\mu_{\min}<\infty,$ $f_{3\min})$ and $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(-\infty<\mu_{\max}<-|f_{3\max}|\,,f_{3\max})$ to obtain hermirreps $\mathcal{D}_{I,F_{12},a_{++0},a_{--0},a_{+-0},a_{-+0}}(f_{3\min}=$ $f_{3\max})$ of $\mathrm{SO}(2)_{F_{12}}\ltimes\mathrm{HW}(2)_{I,a_{++0},a_{--0},a_{+-0},a_{-+0}}$. The vector spaces of these hermirreps are $\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|$ $<\mu_{\min}<\infty,f_{3\min})=\mathcal{H}_{I,F_{12},a_{++0},a_{--0},a_{+-0},a_{-+0}}(f_{3\min}=f_{3\max})=\mathcal{H}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(-\infty<\mu_{\max}<-|f_{3\max}|\,,f_{3\max})$.

For all lowest-weight hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(|f_{3\min}|\le\mu_{\min}<\infty, f_{3\min})$ of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$, the operators of Eq. (4) are given as functions of the operators of Eq. (101) by

$$F_{05} = \mu_{\min}I + a_{++0}a_{--0} + a_{+-0}a_{-+0}, \tag{128}$$

$$F_{12}, \tag{129}$$

$$F_{++0} = a_{++0}\sqrt{\mu_{\min}I + F_{12} + a_{+-0}a_{-+0}}, \tag{130}$$

$$F_{--0} = \sqrt{\mu_{\min}I + F_{12} + a_{+-0}a_{-+0}}\; a_{--0}, \tag{131}$$

$$F_{+-0} = a_{+-0}\sqrt{\mu_{\min}I - F_{12} + a_{++0}a_{--0}}, \tag{132}$$

$$F_{-+0} = \sqrt{\mu_{\min}I - F_{12} + a_{++0}a_{--0}}\; a_{-+0} \tag{133}$$

and the operators of Eq. (1) are given as functions of the operators of Eq. (101) by

$$\begin{aligned}
F_{01} = {} & \tfrac{1}{2}\big(a_{++0}\sqrt{\mu_{\min}I + F_{12} + a_{+-0}a_{-+0}} + \sqrt{\mu_{\min}I + F_{12} + a_{+-0}a_{-+0}}\; a_{--0} \\
& -a_{+-0}\sqrt{\mu_{\min}I - F_{12} + a_{++0}a_{--0}} - \sqrt{\mu_{\min}I - F_{12} + a_{++0}a_{--0}}\; a_{-+0}\big), \\
F_{02} = {} & -\tfrac{i}{2}\big(a_{++0}\sqrt{\mu_{\min}I + F_{12} + a_{+-0}a_{-+0}} - \sqrt{\mu_{\min}I + F_{12} + a_{+-0}a_{-+0}}\; a_{--0} \\
& +a_{+-0}\sqrt{\mu_{\min}I - F_{12} + a_{++0}a_{--0}} - \sqrt{\mu_{\min}I - F_{12} + a_{++0}a_{--0}}\; a_{-+0}\big), \\
F_{05} = {} & \mu_{\min}I + a_{++0}a_{--0} + a_{+-0}a_{-+0}, \\
F_{12}, & \\
F_{15} = {} & -\tfrac{i}{2}\big(a_{++0}\sqrt{\mu_{\min}I + F_{12} + a_{+-0}a_{-+0}} - \sqrt{\mu_{\min}I + F_{12} + a_{+-0}a_{-+0}}\; a_{--0} \\
& -a_{+-0}\sqrt{\mu_{\min}I - F_{12} + a_{++0}a_{--0}} + \sqrt{\mu_{\min}I - F_{12} + a_{++0}a_{--0}}\; a_{-+0}\big), \\
F_{25} = {} & -\tfrac{1}{2}\big(a_{++0}\sqrt{\mu_{\min}I + F_{12} + a_{+-0}a_{-+0}} + \sqrt{\mu_{\min}I + F_{12} + a_{+-0}a_{-+0}}\; a_{--0} \\
& +a_{+-0}\sqrt{\mu_{\min}I - F_{12} + a_{++0}a_{--0}} + \sqrt{\mu_{\min}I - F_{12} + a_{++0}a_{--0}}\; a_{-+0}\big).
\end{aligned} \tag{134}$$

For all highest-weight hermirreps $\mathcal{D}_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}(-\infty<\mu_{\max}\le-|f_{3\max}|, f_{3\max})$ of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$, the operators of Eq. (4) are given as functions of the operators of Eq. (109) by

$$F_{05} = \mu_{\max}I - a_{++0}a_{--0} - a_{+-0}a_{-+0}, \tag{135}$$

$$F_{12}, \tag{136}$$

$$F_{++0} = \sqrt{-\mu_{\max}I - F_{12} + a_{++0}a_{--0}}\; a_{-+0}, \tag{137}$$

$$F_{--0} = a_{+-0}\sqrt{-\mu_{\max}I - F_{12} + a_{++0}a_{--0}}, \tag{138}$$

$$F_{+-0} = \sqrt{-\mu_{\max}I + F_{12} + a_{+-0}a_{-+0}}\; a_{--0}, \tag{139}$$

$$F_{-+0} = a_{++0}\sqrt{-\mu_{\max}I + F_{12} + a_{+-0}a_{-+0}} \tag{140}$$

and the operators of Eq. (1) are given as functions of the operators of Eq. (109) by

$$\begin{aligned}
F_{01} = {} & \tfrac{1}{2}\big(\sqrt{-\mu_{\max}I - F_{12} + a_{++0}a_{--0}}\; a_{-+0} + a_{+-0}\sqrt{-\mu_{\max}I - F_{12} + a_{++0}a_{--0}} \\
& -\sqrt{-\mu_{\max}I + F_{12} + a_{+-0}a_{-+0}}\; a_{--0} - a_{++0}\sqrt{-\mu_{\max}I + F_{12} + a_{+-0}a_{-+0}}\big), \\
F_{02} = {} & -\tfrac{i}{2}\big(\sqrt{-\mu_{\max}I - F_{12} + a_{++0}a_{--0}}\; a_{-+0} - a_{+-0}\sqrt{-\mu_{\max}I - F_{12} + a_{++0}a_{--0}} \\
& +\sqrt{-\mu_{\max}I + F_{12} + a_{+-0}a_{-+0}}\; a_{--0} - a_{++0}\sqrt{-\mu_{\max}I + F_{12} + a_{+-0}a_{-+0}}\big), \\
F_{05} = {} & \mu_{\max}I - a_{++0}a_{--0} - a_{+-0}a_{-+0}, \\
F_{12}, & \\
F_{15} = {} & -\tfrac{i}{2}\big(\sqrt{-\mu_{\max}I - F_{12} + a_{++0}a_{--0}}\; a_{-+0} - a_{+-0}\sqrt{-\mu_{\max}I - F_{12} + a_{++0}a_{--0}} \\
& -\sqrt{-\mu_{\max}I + F_{12} + a_{+-0}a_{-+0}}\; a_{--0} + a_{++0}\sqrt{-\mu_{\max}I + F_{12} + a_{+-0}a_{-+0}}\big), \\
F_{25} = {} & -\tfrac{1}{2}\big(\sqrt{-\mu_{\max}I - F_{12} + a_{++0}a_{--0}}\; a_{-+0} + a_{+-0}\sqrt{-\mu_{\max}I - F_{12} + a_{++0}a_{--0}} \\
& +\sqrt{-\mu_{\max}I + F_{12} + a_{+-0}a_{-+0}}\; a_{--0} + a_{++0}\sqrt{-\mu_{\max}I + F_{12} + a_{+-0}a_{-+0}}\big).
\end{aligned} \tag{141}$$

These are examples of *realization* of a basis of operators in the Lie algebra of $\mathrm{SO}(2,2)_{F_{05},F_{12},F_{01},F_{02},F_{15},F_{25}}$ as functions of a basis of operators in the Lie algebra of $\mathrm{SO}(2)_{F_{12}} \ltimes \mathrm{HW}(2)_{I,a_{++0},a_{--0},a_{+-0},a_{-+0}}$.