Sum of fourth powers of reciprocal natural numbers[1,2]

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Euler[1] divides the infinite power series

$$\sin\theta = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \tag{1}$$

by $y = \sin\theta$ to obtain

$$0 = 1 - \frac{\theta}{1!y} + \frac{\theta^3}{3!y} - \frac{\theta^5}{5!y} + \frac{\theta^7}{7!y} - \cdots = \left(1-\frac{\theta}{\theta_0}\right)\left(1-\frac{\theta}{\theta_1}\right)\left(1-\frac{\theta}{\theta_2}\right)\left(1-\frac{\theta}{\theta_3}\right)\left(1-\frac{\theta}{\theta_4}\right)\cdots, \tag{2}$$

where the (infinitely many) roots θ_0, θ_1, θ_2, θ_3, θ_4, …, θ_n, … are the angles for which $\sin\theta_n = y = \sin\theta$. Equality of coefficients of each power of θ on both sides of the second equality of Eq. (2) gives, for the respective powers θ, θ^2, θ^3, θ^4,

$$\frac{1}{1!y} = \frac{1}{\theta_0}+\frac{1}{\theta_1}+\frac{1}{\theta_2}+\frac{1}{\theta_3}+\frac{1}{\theta_4}+\cdots = \frac{1}{\theta_0}+\frac{1}{\pi-\theta_0}+\frac{1}{-\pi-\theta_0}+\frac{1}{-2\pi+\theta_0}+\frac{1}{2\pi+\theta_0}+\cdots, \tag{3}$$

$$0 = \frac{1}{\theta_0\theta_1}+\frac{1}{\theta_0\theta_2}+\frac{1}{\theta_1\theta_2}+\frac{1}{\theta_0\theta_3}+\frac{1}{\theta_1\theta_3}+\frac{1}{\theta_2\theta_3}+\frac{1}{\theta_0\theta_4}+\frac{1}{\theta_1\theta_4}+\frac{1}{\theta_2\theta_4}+\frac{1}{\theta_3\theta_4}+\cdots, \tag{4}$$

$$-\frac{1}{3!y} = \frac{1}{\theta_0\theta_1\theta_2}+\frac{1}{\theta_0\theta_1\theta_3}+\frac{1}{\theta_0\theta_2\theta_3}+\frac{1}{\theta_1\theta_2\theta_3}+\frac{1}{\theta_0\theta_1\theta_4}+\frac{1}{\theta_0\theta_2\theta_4}+\frac{1}{\theta_1\theta_2\theta_4}+\frac{1}{\theta_0\theta_3\theta_4}+\frac{1}{\theta_1\theta_3\theta_4}+\frac{1}{\theta_2\theta_3\theta_4}+\cdots, \tag{5}$$

$$0 = \frac{1}{\theta_0\theta_1\theta_2\theta_3}+\frac{1}{\theta_0\theta_1\theta_2\theta_4}+\frac{1}{\theta_0\theta_1\theta_3\theta_4}+\frac{1}{\theta_0\theta_2\theta_3\theta_4}+\frac{1}{\theta_1\theta_2\theta_3\theta_4}+\cdots. \tag{6}$$

For the second equality of Eq. (3), the angles θ_n are ordered by magnitude with θ_0 as the angle of smallest magnitude. For $y = \sin\theta = 1$ and $\theta_0 = \pi/2$, pairs of the angles θ_n are equal and Eqs. (3) and (5) reduce to

$$1 = \frac{1}{\frac{\pi}{2}}+\frac{1}{\pi-\frac{\pi}{2}}+\frac{1}{-\pi-\frac{\pi}{2}}+\frac{1}{-2\pi+\frac{\pi}{2}}+\frac{1}{2\pi+\frac{\pi}{2}}+\cdots = \frac{2}{\pi}\left(\frac{1}{1}+\frac{1}{1}-\frac{1}{3}-\frac{1}{3}+\frac{1}{5}+\frac{1}{5}-\frac{1}{7}-\frac{1}{7}+\cdots\right), \tag{7}$$

$$-\frac{1}{6} = \frac{1}{\theta_0\theta_1\theta_2}+\frac{1}{\theta_0\theta_1\theta_3}+\frac{1}{\theta_0\theta_2\theta_3}+\frac{1}{\theta_1\theta_2\theta_3}+\frac{1}{\theta_0\theta_1\theta_4}+\frac{1}{\theta_0\theta_2\theta_4}+\frac{1}{\theta_1\theta_2\theta_4}+\frac{1}{\theta_0\theta_3\theta_4}+\frac{1}{\theta_1\theta_3\theta_4}+\frac{1}{\theta_2\theta_3\theta_4}+\cdots. \tag{8}$$

Euler[1] expresses the sum of fourth powers of terms in an arbitrary series a, b, c, d, e, … as

$$a^4+b^4+c^4+d^4+e^4+\cdots = \alpha^4 - 4\alpha^2\beta + 4\alpha\gamma + 2\beta^2 - 4\delta, \tag{9}$$

where

$$\begin{aligned}\alpha &= a+b+c+d+e+\cdots,\\ \beta &= ab+ac+bc+ad+bd+cd+ae+be+ce+de+\cdots,\\ \gamma &= abc+abd+acd+bcd+abe+ace+bce+ade+bde+cde+\cdots,\\ \delta &= abcd+abce+abde+acde+bcde+\cdots.\end{aligned} \tag{10}$$

For the series in Eq. (7), that is, for

$$a=\frac{1}{\theta_0}=\frac{2}{\pi}\frac{1}{1},\qquad b=\frac{1}{\theta_1}=\frac{2}{\pi}\frac{1}{1},\qquad c=\frac{1}{\theta_2}=-\frac{2}{\pi}\frac{1}{3},\qquad d=\frac{1}{\theta_3}=-\frac{2}{\pi}\frac{1}{3},\qquad e=\frac{1}{\theta_4}=\frac{2}{\pi}\frac{1}{5},\qquad \ldots, \tag{11}$$

α, β, γ, δ of Eq. (10) are given by Eqs. (7), (4), (8), (6), respectively, and Eq. (9) becomes

$$\frac{16}{\pi^4}\left(\frac{1}{1^4}+\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{7^4}+\cdots\right) = 1^4-\frac{4}{6} = \frac{1}{3}. \tag{12}$$

The sum of fourth powers of reciprocal odd natural numbers is $\pi^4/32$ times Eq (12),

$$\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^4} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots = \frac{\pi^4}{96}. \tag{13}$$

The sum of fourth powers of reciprocal natural numbers is 16/15 times Eq (13),[2]

$$\begin{aligned}\sum_{n=1,2,3,\cdots}^{\infty}\frac{1}{n^4} &= \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots = \frac{16}{15}\frac{15}{16}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right) = \frac{16}{15}\left(1-\frac{1}{16}\right)\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)\\ &= \frac{16}{15}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots-\frac{1}{2^4}-\frac{1}{4^4}-\frac{1}{6^4}-\frac{1}{8^4}-\cdots\right) = \frac{16}{15}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots\right) = \frac{16}{15}\frac{\pi^4}{96} = \frac{\pi^4}{90}.\end{aligned} \tag{14}$$

[1] "De Summis Serierum Reciprocarum", Leonhard Euler, *Commentarii Academiae Scientiarum Petropolitanae* **7**, 123-134 (1740). Euler might have presented this sum to the Saint Petersburg Academy of Sciences on 5 December 1735.

[2] This sum is used in integration of Planck's law of radiation to obtain the expression $\sigma \equiv 2\pi^5k^4/\left(15h^3c^2\right)$ for the Stefan-Boltzmann constant, σ, in terms of the speed of light, c, Boltzmann's constant, k, and Planck's constant, h.