Stefan-Boltzmann constant obtained by integration of Planck's law of radiation[1,2,3]

Energy emitted per unit time per unit area is called *radiant emittance*. According to Stefan's law, for a black body in thermodynamic equilibrium at kelvin temperature T, radiant emittance is proportional to T^4,

$$M(T) = \sigma T^4, \tag{1}$$

where σ is a universal constant of proportionality called the *Stefan-Boltzmann constant*.

Radiant emittance $M(T)$ is related to energy density $u(T)$ by

$$M(T) = \frac{c}{4} u(T)\,, \tag{2}$$

where c is the speed of light. Radiant emittance and its frequency and wavelength distributions are related by

$$M(T) = \int_0^\infty d\nu\; M_\nu(T) = \int_0^\infty d\lambda\; M_\lambda(T)\,. \tag{3}$$

Energy density and its frequency and wavelength distributions are related by

$$u(T) = \int_0^\infty d\nu\; u_\nu(T) = \int_0^\infty d\lambda\; u_\lambda(T)\,. \tag{4}$$

Max Planck found that black body data are very well fit by the radiant emittance and energy density frequency and wavelength distributions

$$\begin{aligned} M_\nu(T) &= \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}, & M_\lambda(T) &= \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}, \\ u_\nu(T) &= \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{\frac{h\nu}{kT}}-1}, & u_\lambda(T) &= \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}, \end{aligned} \tag{5}$$

where k is Boltzmann's constant and h is Planck's constant; this is called *Planck's law of radiation*.

The Stefan-Boltzmann constant is obtained by integration of Planck's law of radiation:

$$\begin{aligned} \sigma &= \frac{1}{T^4}M(T) = \frac{1}{T^4}\int_0^\infty d\nu\; M_\nu(T) = \frac{1}{T^4}\int_0^\infty d\nu\; \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1} = \frac{2\pi h}{c^2T^4}\int_0^\infty d\nu\; \frac{\nu^3}{e^{\frac{h\nu}{kT}}-1} = \frac{2\pi k^4}{h^3c^2}\int_0^\infty dx\; \frac{x^3}{e^x-1} \\ &= \frac{2\pi k^4}{h^3c^2}\int_0^\infty dx\; \frac{x^3e^{-x}}{1-e^{-x}} = \frac{2\pi k^4}{h^3c^2}\int_0^\infty dx\; x^3e^{-x}\sum_{n=0}^\infty e^{-nx} = \frac{2\pi k^4}{h^3c^2}\int_0^\infty dx\; x^3\sum_{n=1}^\infty e^{-nx} = \frac{2\pi k^4}{h^3c^2}\sum_{n=1}^\infty\int_0^\infty dx\; x^3e^{-nx} \\ &= -\frac{2\pi k^4}{h^3c^2}\sum_{n=1}^\infty\int_0^\infty dx\; \frac{d^3}{dn^3}e^{-nx} = -\frac{2\pi k^4}{h^3c^2}\sum_{n=1}^\infty\frac{d^3}{dn^3}\int_0^\infty dx\; e^{-nx} = -\frac{2\pi k^4}{h^3c^2}\sum_{n=1}^\infty\frac{d^3}{dn^3}\frac{1}{n} = \frac{2\pi k^4}{h^3c^2}\sum_{n=1}^\infty\frac{6}{n^4} \\ &= \frac{12\pi k^4}{h^3c^2}\sum_{n=1}^\infty\frac{1}{n^4} = \frac{12\pi k^4}{h^3c^2}\frac{\pi^4}{90} = \frac{2\pi^5k^4}{15h^3c^2} \\ &= \frac{2\pi^5}{15}\frac{\left[1.380649\times10^{-23}\text{ kg m}^2/\left(\text{K s}^2\right)\right]^4}{\left(6.62607015\times10^{-34}\text{ kg m}^2/\text{s}\right)^3\left(299792458\text{ m/s}\right)^2} = 5.67037441918442945 39\ldots\times10^{-8}\text{ kg}/\left(\text{K}^4\text{s}^3\right). \end{aligned} \tag{6}$$

The first three equalities use Eqs. (1), (3), and (5), the fifth equality uses the change of variables $x = h\nu/(kT)$, the seventh equality expands the $1-e^{-x}$ denominator as a geometric series of e^{-x}, the fifteenth equality uses the sum $\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$ found by Euler,[4] and the seventeenth equality uses exact expressions for the SI defining constants c, h, and k in terms of the SI base units s, m, kg, and K.[5]

---

[1] Max Planck, *The Theory of Heat Radiation*, Dover, New York 1959. This is a republication of the 1914 English translation by Morton Masius of the 1913 Second Edition.

[2] Arnold Sommerfeld, *Thermodynamics and Statistical Mechanics, Lectures on Theoretical Physics, Volume V*, Academic Press, New York 1956, pages 135-152.

[3] John J. Brehm and William J. Mullin, *Introduction to the Structure of Matter: A Course in Modern Physics*, John Wiley & Sons, New York 1989, ISBN 047160531X, pages 75-99.

[4] "De Summis Serierum Reciprocarum", Leonhard Euler, *Commentarii Academiae Scientiarum Petropolitanae* **7**, 123-134 (1740). Euler might have presented this sum to the Saint Petersburg Academy of Sciences on 5 December 1735.

[5] *The International System of Units (SI)*, 9th edition 2019, Bureau International des Poids et Mesures, Pavillon de Breteuil, F-92312 Sèvres Cedex, France, ISBN 9789282222720, https://www.bipm.org/utils/common/pdf/si-brochure/SI-Brochure-9.pdf.