Preparations, measurements, and predictions of quantum mechanics for 2-level systems

An ideal measurement procedure is mathematically described by a self-adjoint operator $A = A^\dagger$, which, for 2-level systems, may be expressed with respect to a basis of orthogonal unit length vectors $|1\rangle$ and $|2\rangle$ as

$$\begin{aligned} A = IAI &= (|1\rangle\langle 1| + |2\rangle\langle 2|)\, A\, (|1\rangle\langle 1| + |2\rangle\langle 2|) = (|1\rangle \quad |2\rangle)\begin{pmatrix}\langle 1|A|1\rangle & \langle 1|A|2\rangle \\ \langle 2|A|1\rangle & \langle 2|A|2\rangle\end{pmatrix}\begin{pmatrix}\langle 1| \\ \langle 2|\end{pmatrix} \\ &= (|1\rangle \quad |2\rangle)\begin{pmatrix} x_{11} & x_{21} - iy_{21} \\ x_{21} + iy_{21} & x_{22}\end{pmatrix}\begin{pmatrix}\langle 1| \\ \langle 2|\end{pmatrix} = (|1\rangle \quad |2\rangle)\begin{pmatrix} t+z & x-iy \\ x+iy & t-z\end{pmatrix}\begin{pmatrix}\langle 1| \\ \langle 2|\end{pmatrix} = (|1\rangle \quad |2\rangle)(t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3)\begin{pmatrix}\langle 1| \\ \langle 2|\end{pmatrix} \end{aligned} \tag{1}$$

where $x_{11}, x_{21}, y_{21}, x_{22}$ and t, x, y, z are real numbers with the same units as A. The σ_i are called *Pauli matrices*

$$\sigma_0 = I = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}. \tag{2}$$

The result of a single application of an ideal measurement procedure A on an individual 2-level system is described by an eigenvalue of the operator A that describes the measurement procedure. Eigenvalues of A solve the equation

$$\begin{aligned} 0 &= \begin{vmatrix}\lambda - x_{11} & -(x_{21} - iy_{21}) \\ -(x_{21} + iy_{21}) & \lambda - x_{22}\end{vmatrix} = \begin{vmatrix}\lambda - (t+z) & -(x-iy) \\ -(x+iy) & \lambda - (t-z)\end{vmatrix} \\ &= \lambda^2 - (x_{11} + x_{22})\,\lambda - x_{21}^2 - y_{21}^2 + x_{11}x_{22} = \lambda^2 - 2t\lambda + t^2 - x^2 - y^2 - z^2, \end{aligned} \tag{3}$$

are given by

$$\begin{aligned} \lambda_+ &= \tfrac{x_{11}+x_{22}}{2} + \sqrt{x_{21}^2 + y_{21}^2 + \left(\tfrac{x_{11}-x_{22}}{2}\right)^2} = t + \sqrt{x^2+y^2+z^2}, \\ \lambda_- &= \tfrac{x_{11}+x_{22}}{2} - \sqrt{x_{21}^2 + y_{21}^2 + \left(\tfrac{x_{11}-x_{22}}{2}\right)^2} = t - \sqrt{x^2+y^2+z^2}, \end{aligned} \tag{4}$$

and, like the eigenvalues of any self-adjoint operator, are real numbers. Quantum mechanics provides just two real eigenvalues to describe the possible results of a single measurement on an individual 2-level system.

The average value, variance, and standard deviation of the results of $n = n_{\lambda_+} + n_{\lambda_-}$ applications of an ideal measurement procedure A are, respectively,

$$\left(n_{\lambda_+}\lambda_+ + n_{\lambda_-}\lambda_-\right)/n = f_{\lambda_+}\lambda_+ + f_{\lambda_-}\lambda_-, \tag{5}$$

$$\left(n_{\lambda_+}\lambda_+^2 + n_{\lambda_-}\lambda_-^2\right)/n - \left[\left(n_{\lambda_+}\lambda_+ + n_{\lambda_-}\lambda_-\right)/n\right]^2 = f_{\lambda_+}\lambda_+^2 + f_{\lambda_-}\lambda_-^2 - \left(f_{\lambda_+}\lambda_+ + f_{\lambda_-}\lambda_-\right)^2, \tag{6}$$

$$\sqrt{\left(n_{\lambda_+}\lambda_+^2 + n_{\lambda_-}\lambda_-^2\right)/n - \left[\left(n_{\lambda_+}\lambda_+ + n_{\lambda_-}\lambda_-\right)/n\right]^2} = \sqrt{f_{\lambda_+}\lambda_+^2 + f_{\lambda_-}\lambda_-^2 - \left(f_{\lambda_+}\lambda_+ + f_{\lambda_-}\lambda_-\right)^2}, \tag{7}$$

where n_{λ_+} is the number of occurences of λ_+, n_{λ_-} is the number of occurences of λ_-, $f_{\lambda_+} = n_{\lambda_+}/n$ is the frequency of occurence of λ_+, and $f_{\lambda_-} = n_{\lambda_-}/n$ is the frequency of occurence of λ_-. For n large enough so that statistical fluctuations are negligible, the average value, variance, and standard deviation are characteristics of, and depend on, the measurement procedure A combined with a preceding preparation procedure W.

A preparation procedure W is mathematically described by a positive,[1] self-adjoint *statistical operator* $W = W^\dagger$ with unit trace, $\mathrm{Tr}\, W = 1$, which, for 2-level systems, may be expressed as

$$W = (|1\rangle \quad |2\rangle)\begin{pmatrix}\langle 1|W|1\rangle & \langle 1|W|2\rangle \\ \langle 2|W|1\rangle & \langle 2|W|2\rangle\end{pmatrix}\begin{pmatrix}\langle 1| \\ \langle 2|\end{pmatrix} = (|1\rangle \quad |2\rangle)\begin{pmatrix}\frac{1}{2} + \xi & u - iv \\ u + iv & \frac{1}{2} - \xi\end{pmatrix}\begin{pmatrix}\langle 1| \\ \langle 2|\end{pmatrix} \tag{8}$$

where u, v, ξ are unitless real numbers that satisfy

$$u^2 + v^2 + \xi^2 \le \tfrac{1}{4}. \tag{9}$$

The set of statistical operators W which satisfy Eq. (9) is a solid 3-ball of radius $\frac{1}{2}$.

The average value, variance, and standard deviation of the results of applications of a preparation procedure W followed by an ideal measurement procedure A are mathematically described by, respectively,

$$\mathrm{Tr}\,(AW) = t + 2\,(xu + yv + z\xi)\,, \tag{10}$$

$$\mathrm{Tr}\left(\left(A - \mathrm{Tr}\,(AW)\, I\right)^2 W\right) = \mathrm{Tr}\left(A^2 W\right) - \mathrm{Tr}^2(AW) = x^2 + y^2 + z^2 - 4\,(xu + yv + z\xi)^2\,, \tag{11}$$

$$\sqrt{\mathrm{Tr}\left(\left(A - \mathrm{Tr}\,(AW)\, I\right)^2 W\right)} = \sqrt{\mathrm{Tr}\,(A^2 W) - \mathrm{Tr}^2(AW)} = \sqrt{x^2 + y^2 + z^2 - 4\,(xu + yv + z\xi)^2}. \tag{12}$$

$\mathrm{Tr}\,(AW)$ denotes the trace of the product of the operator A with the statistical operator W. The quantities in Eqs. (10), (11), and (12) are the predictions of quantum mechanics for the average value, variance, and standard deviation and are called the *expectation value*, *dispersion*, and *uncertainty*, respectively.

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[1] An operator W is called *positive* if $\langle\phi|\, W\, |\phi\rangle$ is real and non-negative for all vectors $|\phi\rangle = c_1\,|1\rangle + c_2\,|2\rangle$, where c_1 and c_2 are arbitrary complex numbers.

For an arbitrary self-adjoint operator A, that is, for arbitrary but fixed values of t, x, y, z in Eq. (1), the only two statistical operators for which dispersion and uncertainty are zero are those for which u, v, ξ in Eq. (8) are given by

$$u = \frac{x}{2\sqrt{x^2+y^2+z^2}}, \qquad v = \frac{y}{2\sqrt{x^2+y^2+z^2}}, \qquad \xi = \frac{z}{2\sqrt{x^2+y^2+z^2}} \tag{13}$$

or by

$$u = -\frac{x}{2\sqrt{x^2+y^2+z^2}}, \qquad v = -\frac{y}{2\sqrt{x^2+y^2+z^2}}, \qquad \xi = -\frac{z}{2\sqrt{x^2+y^2+z^2}}. \tag{14}$$

The expectation values of A with these statistical operators equal the respective eigenvalues λ_+ and λ_- of A given in Eq. (4) and describe average values in Eq. (5) for which the frequencies of occurence of results described by the eigenvalues are given, respectively, by

$$f_{\lambda_+} = 1 \quad \text{and} \quad f_{\lambda_-} = 0 \tag{15}$$

and by

$$f_{\lambda_+} = 0 \quad \text{and} \quad f_{\lambda_-} = 1. \tag{16}$$

The statistical operators which fulfill Eq. (13) or Eq. (14) fulfill the equality in Eq. (9) and are antipodes on the 2-sphere surface of the 3-ball of radius $\frac{1}{2}$.

For an arbitrary self-adjoint operator A, the statistical operators for which dispersion and uncertainty are maximum are those for which u, v, ξ satisfy

$$xu + yv + z\xi = 0. \tag{17}$$

The expectation value of A with any of these statistical operators equals the average of the eigenvalues, $t = (\lambda_+ + \lambda_-)/2$, and describes an average value in Eq. (5) in which the frequencies of occurence of results described by the eigenvalues are equal,

$$f_{\lambda_+} = \tfrac{1}{2} \quad \text{and} \quad f_{\lambda_-} = \tfrac{1}{2}. \tag{18}$$

The set of these statistical operators is a disk of radius $\frac{1}{2}$ concentric with the 3-ball of radius $\frac{1}{2}$ and which perpendicularly bisects the line segment that connects the antipodal points given by Eqs. (13) and (14).

Experimental estimation of all of the parameters u, v, ξ of the statistical operator W that describes an arbitrary preparation procedure W involves application of the preparation procedure separately followed by (at least) three different measurement procedures. A convenient set of three measurement procedures X, Y, Z is those mathematically described by the operators

$$X = (|1\rangle \quad |2\rangle)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix}, \quad Y = (|1\rangle \quad |2\rangle)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix}, \quad Z = (|1\rangle \quad |2\rangle)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix}, \tag{19}$$

which have the respective pairs of eigenvalues

$$x_+ = 1,\ x_- = -1, \qquad y_+ = 1,\ y_- = -1, \qquad z_+ = 1,\ z_- = -1. \tag{20}$$

The expectation values of the statistical operator W with the operators X, Y, Z are

$$\mathrm{Tr}\,(XW) = 2u, \qquad \mathrm{Tr}\,(YW) = 2v, \qquad \mathrm{Tr}\,(ZW) = 2\xi. \tag{21}$$

Estimates of the parameters u, v, ξ are given, according to Eqs. (5), (20), and (21), by the respective frequency of occurance half-differences $\left(f_{x_+} - f_{x_-}\right)/2$, $\left(f_{y_+} - f_{y_-}\right)/2$, $\left(f_{z_+} - f_{z_-}\right)/2$ of results of the measurement procedures X, Y, Z preceded by the preparation procedure W.

Experimental estimation of all of the parameters t, x, y, z of the self-adjoint operator A that describes an arbitrary measurement procedure A involves application of the measurement procedure separately preceded by (at least) three different preparation procedures. A convenient set of three preparation procedures U, V, Ξ is those mathematically described by the statistical operators

$$U = (|1\rangle \quad |2\rangle)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix}, \quad V = (|1\rangle \quad |2\rangle)\begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix}, \quad \Xi = (|1\rangle \quad |2\rangle)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix}. \tag{22}$$

The expectation values of the operator A with the statistical operators U, V, Ξ are

$$\mathrm{Tr}\,(AU) = t + x, \qquad \mathrm{Tr}\,(AV) = t + y, \qquad \mathrm{Tr}\,(A\Xi) = t + z. \tag{23}$$

The eigenvalues λ_+ and λ_- of operator A are estimated from the results of all applications of measurement procedure A, with no regard to preparation procedures. The estimate of t is, according to Eq. (4), half the sum of the estimates of the eigenvalues,

$$t = (\lambda_+ + \lambda_-)/2. \tag{24}$$

Estimates of $t + x$, $t + y$, $t + z$ are given by Eq. (5) in which the freqencies of occurence f_{λ_+} and f_{λ_-} depend on which of the preparation procedures U, V, Ξ precedes the measurement procedure A. Estimates of x, y, z are given by subtraction of the estimate of t from the estimates of $t + x$, $t + y$, $t + z$.