**Chronometric time evolution, redshift of light, and galaxy GN-z11**

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Einstein[1] proposes that, on the scale of the universe, 3-dimensional space approximates a 3-sphere of radius $R$ and volume $2\pi^2R^3$. Segal[2] proposes that the group of causality preserving transformations of Einstein space-time is the universal covering group of the group $\mathrm{SO}(2,4)_{F_{ab}}$. The (complexified) Lie algebra of $\mathrm{SO}(2,4)_{F_{ab}}$ has a basis of fifteen operators,

$$\begin{array}{ccccc} F_{01}, & F_{02}, & F_{03}, & F_{04}, & F_{05}, \\ & F_{12}, & F_{13}, & F_{14}, & F_{15}, \\ & & F_{23}, & F_{24}, & F_{25}, \\ & & & F_{34}, & F_{35}, \\ & & & & F_{45}, \end{array} \tag{1}$$

that satisfy the commutation relations

$$[F_{ab},F_{cd}] \equiv F_{ab}F_{cd} - F_{cd}F_{ab} = -i\left(\eta_{ac}F_{bd} + \eta_{bd}F_{ac} - \eta_{ad}F_{bc} - \eta_{bc}F_{ad}\right), \qquad \text{for } a,b \in \{0,1,2,3,4,5\}, \tag{2}$$

where $F_{ab} = -F_{ba}$, $\eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = -\eta_{44} = \eta_{55} = 1$, and $\eta_{ab} = 0$ for $a \neq b$. The commutators $[F_{05},F_{04}] = iF_{45}$ and $[F_{05},F_{45}] = -iF_{04}$ may be used to compute the nested commutator

$$\underbrace{[F_{05},\cdots[F_{05},}_{n}F_{04}]\cdots] = \begin{cases} F_{04}, & \text{for } n \in \{0,2,4,\ldots\}, \\ iF_{45}, & \text{for } n \in \{1,3,5,\ldots\}. \end{cases} \tag{3}$$

Segal[3] proposes that time evolution of an operator $A$ is given by

$$A(t) = e^{-itH/\hbar}A(0)\,e^{itH/\hbar} = \sum_{n=0}^{\infty}\frac{(-it/\hbar)^n}{n!}\underbrace{[H,\cdots[H,}_{n}A(0)]\cdots], \tag{4}$$

where the operator $H$ which governs time evolution is

$$H = \tfrac{hc}{2\pi R}F_{05} = \tfrac{\hbar c}{R}F_{05}. \tag{5}$$

The operator $H$ is called the *Einstein energy operator* and the parameter $t$ is called the *Einstein time*. The Einstein energy operator commutes with itself and does not change with Einstein time,

$$H(t) = e^{-itH/\hbar}H(0)\,e^{itH/\hbar} = H(0) = H. \tag{6}$$

Segal proposes that Minkowski energy is represented by the operator

$$H_0(0) = \tfrac{\hbar c}{2R}(F_{05} - F_{04}) \tag{7}$$

and that gravitational energy is represented by the operator

$$H_1(0) = H - H_0(0) = \tfrac{\hbar c}{2R}(F_{05} + F_{04}). \tag{8}$$

The operators $H_0(0)$ and $H_1(0)$ do not commute with the Einstein energy operator $H$,

$$[H,H_0(0)] = \left[\tfrac{\hbar c}{R}F_{05}, \tfrac{\hbar c}{2R}(F_{05}-F_{04})\right] = -\tfrac{\hbar^2c^2}{2R^2}[F_{05},F_{04}] = -i\tfrac{\hbar^2c^2}{2R^2}F_{45}, \qquad [H,H_1(0)] = i\tfrac{\hbar^2c^2}{2R^2}F_{45}, \tag{9}$$

and evolve with Einstein time $t$ in accord with

$$\begin{aligned} H_0(t) = e^{-itH/\hbar}H_0(0)e^{itH/\hbar} &= e^{-i\frac{ct}{R}F_{05}}\tfrac{\hbar c}{2R}(F_{05}-F_{04})\,e^{i\frac{ct}{R}F_{05}} = \tfrac{\hbar c}{2R}F_{05} - \tfrac{\hbar c}{2R}e^{-i\frac{ct}{R}F_{05}}F_{04}e^{i\frac{ct}{R}F_{05}} \\ &= \tfrac{\hbar c}{2R}F_{05} - \tfrac{\hbar c}{2R}\textstyle\sum_{n=0}^{\infty}\frac{(-ict/R)^n}{n!}\underbrace{[F_{05},\cdots[F_{05},}_{n}F_{04}]\cdots] \\ &= \tfrac{\hbar c}{2R}F_{05} - \tfrac{\hbar c}{2R}F_{04}\textstyle\sum_{n=0,2,4,\cdots}\frac{1}{n!}(-1)^{\frac{n}{2}}(ct/R)^n + \tfrac{\hbar c}{2R}F_{45}\sum_{n=1,3,5,\cdots}\frac{1}{n!}(-1)^{\frac{n-1}{2}}(ct/R)^n \\ &= \tfrac{\hbar c}{2R}F_{05} - \cos\!\left(\tfrac{ct}{R}\right)\tfrac{\hbar c}{2R}F_{04} - \sin\!\left(\tfrac{ct}{R}\right)\tfrac{\hbar c}{2R}F_{45} \\ &= \tfrac{1}{2}\left[1+\cos\!\left(\tfrac{ct}{R}\right)\right]\tfrac{\hbar c}{2R}(F_{05}-F_{04}) + \tfrac{1}{2}\left[1-\cos\!\left(\tfrac{ct}{R}\right)\right]\tfrac{\hbar c}{2R}(F_{05}+F_{04}) - \sin\!\left(\tfrac{ct}{R}\right)\tfrac{\hbar c}{2R}F_{45} \\ &= \tfrac{1}{2}\left[1+\cos\!\left(\tfrac{ct}{R}\right)\right]H_0(0) + \tfrac{1}{2}\left[1-\cos\!\left(\tfrac{ct}{R}\right)\right]H_1(0) - \sin\!\left(\tfrac{ct}{R}\right)\tfrac{\hbar c}{2R}F_{45}, \end{aligned} \tag{10}$$

$$\begin{aligned} H_1(t) = e^{-itH/\hbar}H_1(0)e^{itH/\hbar} &= H - H_0(t) = \tfrac{\hbar c}{R}F_{05} - \tfrac{\hbar c}{2R}F_{05} + \cos\!\left(\tfrac{ct}{R}\right)\tfrac{\hbar c}{2R}F_{04} + \sin\!\left(\tfrac{ct}{R}\right)\tfrac{\hbar c}{2R}F_{45} \\ &= \tfrac{1}{2}\left[1-\cos\!\left(\tfrac{ct}{R}\right)\right]\tfrac{\hbar c}{2R}(F_{05}-F_{04}) + \tfrac{1}{2}\left[1+\cos\!\left(\tfrac{ct}{R}\right)\right]\tfrac{\hbar c}{2R}(F_{05}+F_{04}) + \sin\!\left(\tfrac{ct}{R}\right)\tfrac{\hbar c}{2R}F_{45} \\ &= \tfrac{1}{2}\left[1-\cos\!\left(\tfrac{ct}{R}\right)\right]H_0(0) + \tfrac{1}{2}\left[1+\cos\!\left(\tfrac{ct}{R}\right)\right]H_1(0) + \sin\!\left(\tfrac{ct}{R}\right)\tfrac{\hbar c}{2R}F_{45}. \end{aligned} \tag{11}$$

Einstein energy, which is conserved, is the sum of Minkowski energy and gravitational energy, which are not separately conserved.

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[1] "Cosmological considerations on the general theory of relativity", A. Einstein, in *The principle of relativity*, Dover, New York 1952, pages 177-188, translated from "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie", A. Einstein, *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, 1917, pages 142-152.

[2] Irving Ezra Segal, *Mathematical cosmology and extragalactic astronomy*, Academic Press, New York, 1976, Page 58.

[3] Irving Ezra Segal, *Mathematical cosmology and extragalactic astronomy*, Academic Press, New York, 1976, Page 76.

Let $|x\rangle = |x_0\,\mathbf{x}\rangle$ be a basis of generalized eigenvectors labeled by Minkowski time $x_0$ and Minkowski position $\mathbf{x}$. For a state represented by a highly localized wavefunction $\psi(x) = \langle x|\psi\rangle$, an intrinsic spinless realization of the $F_{ab}$ (which neglects photon polarization) is given by

$$
\begin{aligned}
\langle x|F_{01}|\psi\rangle &= i\Big(cx_0\tfrac{\partial}{\partial x_1}+\tfrac{x_1}{c}\tfrac{\partial}{\partial x_0}\Big)\langle x|\psi\rangle, \quad \langle x|F_{02}|\psi\rangle = i\Big(cx_0\tfrac{\partial}{\partial x_2}+\tfrac{x_2}{c}\tfrac{\partial}{\partial x_0}\Big)\langle x|\psi\rangle, \quad \langle x|F_{03}|\psi\rangle = i\Big(cx_0\tfrac{\partial}{\partial x_3}+\tfrac{x_3}{c}\tfrac{\partial}{\partial x_0}\Big)\langle x|\psi\rangle,\\
\langle x|F_{12}|\psi\rangle &= -i\Big(x_1\tfrac{\partial}{\partial x_2}-x_2\tfrac{\partial}{\partial x_1}\Big)\langle x|\psi\rangle, \quad \langle x|F_{13}|\psi\rangle = -i\Big(x_1\tfrac{\partial}{\partial x_3}-x_3\tfrac{\partial}{\partial x_1}\Big)\langle x|\psi\rangle,\\
\langle x|F_{23}|\psi\rangle &= -i\Big(x_2\tfrac{\partial}{\partial x_3}-x_3\tfrac{\partial}{\partial x_2}\Big)\langle x|\psi\rangle,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\langle x|F_{04}|\psi\rangle &= i\Big\{\big[1+\tfrac{1}{4R^2}\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\big]\tfrac{R}{c}\tfrac{\partial}{\partial x_0}-\tfrac{c}{2R}x_0\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\Big\}\langle x|\psi\rangle,\\
\langle x|F_{14}|\psi\rangle &= i\Big\{\big[1+\tfrac{1}{4R^2}\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\big]R\tfrac{\partial}{\partial x_1}+\tfrac{1}{2R}x_1\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\Big\}\langle x|\psi\rangle,\\
\langle x|F_{24}|\psi\rangle &= i\Big\{\big[1+\tfrac{1}{4R^2}\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\big]R\tfrac{\partial}{\partial x_2}+\tfrac{1}{2R}x_2\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\Big\}\langle x|\psi\rangle,\\
\langle x|F_{34}|\psi\rangle &= i\Big\{\big[1+\tfrac{1}{4R^2}\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\big]R\tfrac{\partial}{\partial x_3}+\tfrac{1}{2R}x_3\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\Big\}\langle x|\psi\rangle,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\langle x|F_{05}|\psi\rangle &= i\Big\{-\big[1-\tfrac{1}{4R^2}\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\big]\tfrac{R}{c}\tfrac{\partial}{\partial x_0}-\tfrac{c}{2R}x_0\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\Big\}\langle x|\psi\rangle,\\
\langle x|F_{15}|\psi\rangle &= i\Big\{-\big[1-\tfrac{1}{4R^2}\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\big]R\tfrac{\partial}{\partial x_1}+\tfrac{1}{2R}x_1\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\Big\}\langle x|\psi\rangle,\\
\langle x|F_{25}|\psi\rangle &= i\Big\{-\big[1-\tfrac{1}{4R^2}\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\big]R\tfrac{\partial}{\partial x_2}+\tfrac{1}{2R}x_2\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\Big\}\langle x|\psi\rangle,\\
\langle x|F_{35}|\psi\rangle &= i\Big\{-\big[1-\tfrac{1}{4R^2}\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\big]R\tfrac{\partial}{\partial x_3}+\tfrac{1}{2R}x_3\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\Big\}\langle x|\psi\rangle,
\end{aligned}
\tag{14}
$$

$$
\langle x|F_{45}|\psi\rangle = i\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\langle x|\psi\rangle. \tag{15}
$$

Minkowski energy and momenta have the intrinsic spinless realizations

$$
\begin{aligned}
\langle x|H_0(0)|\psi\rangle &= \langle x|\tfrac{\hbar c}{2R}(F_{05}-F_{04})|\psi\rangle = -i\hbar\tfrac{\partial}{\partial x_0}\langle x|\psi\rangle,\\
\langle x|P_1(0)|\psi\rangle &= \langle x|\tfrac{\hbar}{2R}(F_{15}-F_{14})|\psi\rangle = -i\hbar\tfrac{\partial}{\partial x_1}\langle x|\psi\rangle,\\
\langle x|P_2(0)|\psi\rangle &= \langle x|\tfrac{\hbar}{2R}(F_{25}-F_{24})|\psi\rangle = -i\hbar\tfrac{\partial}{\partial x_2}\langle x|\psi\rangle,\\
\langle x|P_3(0)|\psi\rangle &= \langle x|\tfrac{\hbar}{2R}(F_{35}-F_{34})|\psi\rangle = -i\hbar\tfrac{\partial}{\partial x_3}\langle x|\psi\rangle.
\end{aligned}
\tag{16}
$$

Gravitational energy and momenta have the intrinsic spinless realizations

$$
\begin{aligned}
\langle x|\tfrac{\hbar c}{2R}(F_{05}+F_{04})|\psi\rangle &= i\tfrac{\hbar}{4R^2}\Big[\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\tfrac{\partial}{\partial x_0}-2c^2x_0\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\Big]\langle x|\psi\rangle,\\
\langle x|\tfrac{\hbar}{2R}(F_{15}+F_{14})|\psi\rangle &= i\tfrac{\hbar}{4R^2}\Big[\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\tfrac{\partial}{\partial x_1}+2x_1\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\Big]\langle x|\psi\rangle,\\
\langle x|\tfrac{\hbar}{2R}(F_{25}+F_{24})|\psi\rangle &= i\tfrac{\hbar}{4R^2}\Big[\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\tfrac{\partial}{\partial x_2}+2x_2\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\Big]\langle x|\psi\rangle,\\
\langle x|\tfrac{\hbar}{2R}(F_{35}+F_{34})|\psi\rangle &= i\tfrac{\hbar}{4R^2}\Big[\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\tfrac{\partial}{\partial x_3}+2x_3\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\Big]\langle x|\psi\rangle.
\end{aligned}
\tag{17}
$$

Matrix elements of the Minkowski energy operator evolve with Einstein time $t$ in accord with

$$
\begin{aligned}
\langle x|H_0(t)|\psi\rangle &= \tfrac{1}{2}\big[1+\cos\big(\tfrac{ct}{R}\big)\big]\langle x|H_0(0)|\psi\rangle + \tfrac{1}{2}\big[1-\cos\big(\tfrac{ct}{R}\big)\big]\langle x|H_1(0)|\psi\rangle - \sin\big(\tfrac{ct}{R}\big)\tfrac{\hbar c}{2R}\langle x|F_{45}|\psi\rangle\\
&= \tfrac{1}{2}\big[1+\cos\big(\tfrac{ct}{R}\big)\big]\langle x|H_0(0)|\psi\rangle + \tfrac{1}{2}\big[1-\cos\big(\tfrac{ct}{R}\big)\big]\tfrac{\hbar c}{2R}\langle x|(F_{05}+F_{04})|\psi\rangle - \sin\big(\tfrac{ct}{R}\big)\tfrac{\hbar c}{2R}\langle x|F_{45}|\psi\rangle\\
&= \tfrac{1}{2}\big[1+\cos\big(\tfrac{ct}{R}\big)\big]\langle x|H_0(0)|\psi\rangle - \tfrac{1}{2}\big[1-\cos\big(\tfrac{ct}{R}\big)\big]\tfrac{1}{4R^2}\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\langle x|H_0(0)|\psi\rangle\\
&\qquad - \big\{\big[1-\cos\big(\tfrac{ct}{R}\big)\big]\tfrac{c}{2R}x_0+\sin\big(\tfrac{ct}{R}\big)\big\}\tfrac{\hbar c}{2R}\langle x|F_{45}|\psi\rangle,
\end{aligned}
\tag{18}
$$

where the first equality is the matrix element of Eq. (10) between $\langle x|$ and $|\psi\rangle$, the second equality uses Eq. (8), and the third equality uses the intrinsic spinless realization to express the sum of $\langle x|F_{05}|\psi\rangle$ from Eq. (14) with $\langle x|F_{04}|\psi\rangle$ from Eq. (13) in terms of $\langle x|H_0(0)|\psi\rangle$ from Eq. (16) and $\langle x|F_{45}|\psi\rangle$ from Eq. (15).

Consider emission of light in a state $|\nu_{\rm emit}\rangle$ with sharp value $2\pi\hbar\nu_{\rm emit}$ of Minkowski energy,

$$
H_0(0)|\nu_{\rm emit}\rangle \simeq |\nu_{\rm emit}\rangle\,2\pi\hbar\nu_{\rm emit}. \tag{19}
$$

The Einstein time evolved Minkowski energy matrix element $\langle x|H_0(t)|\nu_{\rm emit}\rangle$ is, from Eqs. (18), (19), and (15), given by

$$
\begin{aligned}
\langle x|H_0(t)|\nu_{\rm emit}\rangle &\simeq \langle x|\nu_{\rm emit}\rangle\,\tfrac{1}{2}\big[1+\cos\big(\tfrac{ct}{R}\big)\big]\,2\pi\hbar\nu_{\rm emit}\\
&\quad - \langle x|\nu_{\rm emit}\rangle\,\tfrac{1}{2}\big[1-\cos\big(\tfrac{ct}{R}\big)\big]\tfrac{1}{4R^2}\big(c^2x_0^2-x_1^2-x_2^2-x_3^2\big)\,2\pi\hbar\nu_{\rm emit}\\
&\quad - i\big\{\big[1-\cos\big(\tfrac{ct}{R}\big)\big]\tfrac{c}{2R}x_0+\sin\big(\tfrac{ct}{R}\big)\big\}\tfrac{\hbar c}{2R}\Big(x_0\tfrac{\partial}{\partial x_0}+x_1\tfrac{\partial}{\partial x_1}+x_2\tfrac{\partial}{\partial x_2}+x_3\tfrac{\partial}{\partial x_3}\Big)\langle x|\nu_{\rm emit}\rangle\,.
\end{aligned}
\tag{20}
$$

Along any light ray, $c^2x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0$ and the second term in Eq. (20) is zero. Segal[4] considers wave functions (such as plane waves) for which the third term in Eq. (20) is also negligible so that, to good approximation,

$$\langle x|H_0(t)|\nu_{\rm emit}\rangle \simeq \langle x\,|\nu_{\rm emit}\rangle\, \tfrac{1}{2}\big[1+\cos\big(\tfrac{ct}{R}\big)\big]\, 2\pi\hbar\nu_{\rm emit} \simeq \langle x\,|\nu_{\rm emit}\rangle\, 2\pi\hbar\nu_{\rm obs}(t)\,, \tag{21}$$

where

$$\nu_{\rm obs}(t) \simeq \tfrac{1}{2}\big[1+\cos\big(\tfrac{ct}{R}\big)\big]\,\nu_{\rm emit} = \tfrac{1}{2}(1+\cos\theta)\,\nu_{\rm emit} \tag{22}$$

is the observed frequency at Einstein time $t$, great circle distance $ct$, and great circle angle $\theta = ct/R$.

The redshift of this light is, to good approximation, given by

$$\begin{aligned} z \equiv \text{ redshift } &\equiv \frac{\text{emitted frequency } - \text{ observed frequency}}{\text{observed frequency}} \\ &\equiv \frac{\nu_{\rm emit}-\nu_{\rm obs}}{\nu_{\rm obs}} = \frac{1-\frac{\nu_{\rm obs}}{\nu_{\rm emit}}}{\frac{\nu_{\rm obs}}{\nu_{\rm emit}}} \simeq \frac{1-\frac{1}{2}(1+\cos\theta)}{\frac{1}{2}(1+\cos\theta)} = \frac{1-\cos\theta}{1+\cos\theta} = \tan^2\!\big(\tfrac{\theta}{2}\big). \end{aligned} \tag{23}$$

This *chronometric redshift* of light depends on the great circle angular separation $\theta = ct/R$ of the source from the observer. Chronometric redshift does not require any assumption that the source moves with respect to the observer; if the source has some peculiar motion toward or away from the observer, then chronometric redshift is modified by some Doppler blueshift or Doppler redshift.

The fraction of an Einstein universe's 3-sphere spatial volume $2\pi^2R^3$ that is beyond a great circle angle $\theta$, or beyond a purely chronometric redshift $z$, is given by

$$R^3\!\int_\theta^\pi\! d\theta' \sin^2\theta' \int_0^\pi\! d\psi\ \sin\psi \int_0^{2\pi}\! d\varphi \Big/ \big(2\pi^2R^3\big) = 1-\tfrac{1}{2\pi}[2\theta-\sin(2\theta)] \simeq 1-\tfrac{2}{\pi}\Big[\arctan\big(\sqrt{z}\big)-\tfrac{\sqrt{z}(1-z)}{(1+z)^2}\Big]\,. \tag{24}$$

GN-$z$11 is a galaxy near the Big Dipper from which light is received with a redshift of $z = 11.09$.[5] If this redshift is purely chronometric (negligible peculiar motion), then the great circle angular separation of GN-$z$11 from us is

$$\theta \simeq 2\arctan\big(\sqrt{z}\big) = 2\arctan\Big(\sqrt{11.09}\Big) = 2.558 = 146.6^\circ, \tag{25}$$

the great circle angular separation of GN-$z$11 from the farthest point from us in the universe (our antipode) is $0.583 = \pi - 2.5581554 = 180^\circ - 146.571509^\circ = 33.4^\circ$, and **less than 4 percent** of the universe is farther than GN-$z$11 from us.

If the radius of the universe is $R = 504.6 \pm 126$ million lightyears,[6] then the great circle distance of GN-$z$11 from us is roughly

$$R\theta \simeq (504.5648 \text{ million lightyears})(2.5581554) = 1.29 \text{ billion lightyears} \tag{26}$$

and the great circle distance of GN-$z$11 from the farthest point from us in the universe (our antipode) is roughly (504.5648 million lightyears)(0.5834372) = 294 million lightyears. This estimate of the distance of GN-$z$11 from us is much lower than other distance estimates.[7]

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[4] Irving Ezra Segal, *Mathematical cosmology and extragalactic astronomy*, Academic Press, New York, 1976, pages 78-82. Segal also estimates that the standard deviation of observed frequency is negligible for a discrete emitted frequency equal to or higher than that of the 21 cm spectral line of atomic Hydrogen.

[5] "A remarkably luminous galaxy at $z = 11.1$ measured with *Hubble Space Telescope* grism spectroscopy", P. A. Oesch, G. Brammer, P. G. van Dokkum, G. D. Illingworth, R. J. Bouwens, I. Labbé, M. Franx, I. Momcheva, M. L. N. Ashby, G. G. Fazio, V. Gonzalez, B. Holden, D. Magee, R. E. Skelton, R. Smit, L. R. Spitler, M. Trenti, S. P. Willner, *The Astrophysical Journal* **819**:129, 1-11 (2016), https://iopscience.iop.org/article/10.3847/0004-637X/819/2/129/pdf.

[6] "Apparent superluminal sources, comparative cosmology and the cosmic distance scale", I. E. Segal, *Monthly Notices of the Royal Astronomical Society* **242**, 423-427 (1990), gives the estimate $R = 160 \pm 40$ million parsecs. My fit to the same observed data gives $R \simeq 154.7$ million parsecs $\simeq 504.6$ million lightyears.

[7] For example, stereographic projection of a line from our antipode through GN-$z$11 onto a Minkowski (flat) space-time that is tangent to our (curved) Einstein universe at our point of observation gives an image of GN-$z$11 in Minkowski space-time with projected Minkowski distance from us

$$d = 2R\tan\big(\tfrac{\theta}{2}\big) \simeq 2R\sqrt{z} = 2\,(504.5648 \text{ million lightyears})\sqrt{11.09} = 3.36 \text{ billion lightyears} \tag{27}$$

and with projected Minkowski time for light to travel to us

$$x_0 = \tfrac{2R}{c}\tan\big(\tfrac{ct}{2R}\big) \simeq \tfrac{2R}{c}\sqrt{z} = 2\,(504.5648 \text{ million years})\sqrt{11.09} = 3.36 \text{ billion years.} \tag{28}$$

If this stereographically projected image of GN-$z$11 is taken to be real, instead of imaginary, then Eq. (27) gives an estimate for the distance of GN-$z$11 from us which is 2.60 times larger than the estimate given in Eq. (26).

The current distance of the Voyager I spacecraft from us is $ct = 22.4$ trillion meters; with $\theta = ct/R$ and $R \simeq 504.6$ Mly, the first equality of Eq. (27) gives a projected Minkowski distance $d$ that is farther by only about 41.2 picometers.

A projected Minkowski time $x_0$ of about 13.8 billion years is given by Eq. (28) for a great circle angle of about $\theta = ct/R = 2.9956 = 171.6^\circ$ and an Einstein time of roughly $t = 1.51$ billion years (roughly one-third the age of Earth).