# Advance of the perihelion of Mercury

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The sidereal period, semimajor axis, and eccentricity of Mercury's orbit around the Sun are roughly[1]

$$\begin{aligned} T &= 87.96935070323211 \text{ days} = 7600551.900759254 \text{ s}, \\ a &= 0.3870989533892468 \text{ AU} = 57909179177.22987 \text{ m}, \\ e &= 0.2056247170222869. \end{aligned} \tag{1}$$

The perihelion of Mercury's orbit advances at an observed rate of roughly $5603.33 \pm 2.11\mathcal{T}$ ″/century, where $\mathcal{T}$ is time in Julian centuries between J2000.0 and the time of interest.[2] Of this, roughly $5029.10 \pm 2.22\mathcal{T}$ ″/century is due to precession of Earth's equinoxes,[2] roughly 2.30 ″/century is due to motion of the ecliptic,[3] roughly $0.0286 \pm 0.0011$ ″/century is due to solar oblateness,[4] and roughly $532.3035 \pm 0.001$ ″/century is due to influences of planets and asteroids on Mercury's orbit as predicted by Newton's theory of gravity.[4] **We wish to account for the remaining roughly 39.5979 ″/century of Mercury's precession.**

For a central potential of the form[5]

$$V(r) = E - \frac{L^2}{2\mu r^2} - \frac{L^2}{2\mu}\left(A + \frac{B}{r} + \frac{C}{r^2} + \frac{D}{r^3} + \frac{\mathcal{E}}{r^4} + \cdots\right), \tag{2}$$

or, in terms of inverse distance $u = 1/r$, of the form

$$V\left(\tfrac{1}{u}\right) = E - \frac{L^2}{2\mu}u^2 - \frac{L^2}{2\mu}\left(A + Bu + Cu^2 + Du^3 + \mathcal{E}u^4 + \cdots\right), \tag{3}$$

the differential equation for the orbit[6] is

$$d\theta = -\frac{du}{\sqrt{\frac{2\mu E}{L^2} - u^2 - \frac{2\mu V}{L^2}}} = -\frac{du}{\sqrt{A + Bu + Cu^2 + Du^3 + \mathcal{E}u^4 + \cdots}} \tag{4}$$

and inverse distance turning points are roots of the equation

$$0 = A + Bu + Cu^2 + Du^3 + \mathcal{E}u^4 + \cdots. \tag{5}$$

If $C \neq 0$ and the coefficients of terms beyond quadratic in $u$ are zero, then Eq. (5) simplifies to a quadratic equation,

$$0 = A + Bu + Cu^2 = C(u - u_{\rm per})(u - u_{\rm ap}), \tag{6}$$

the ratios $A/C$ and $B/C$ are given in terms of the roots $u_{\rm per}$ and $u_{\rm ap}$ by

$$A/C = u_{\rm per}u_{\rm ap}, \qquad B/C = -\left(u_{\rm per} + u_{\rm ap}\right), \tag{7}$$

and the roots are given by the quadratic formula,

$$\frac{1}{r_{\rm per}} = u_{\rm per} = \frac{-B - \sqrt{B^2 - 4AC}}{2C}, \qquad \frac{1}{r_{\rm ap}} = u_{\rm ap} = \frac{-B + \sqrt{B^2 - 4AC}}{2C}. \tag{8}$$

The types of roots are determined by the value of the discriminant

$$\Delta_2 \equiv \left(u_{\rm per} - u_{\rm ap}\right)^2 = \left(B^2 - 4AC\right)/C^2. \tag{9}$$

If $\Delta_2 > 0$, then the roots are real and unequal; if $\Delta_2 = 0$, then the roots are real and equal; and if $\Delta_2 < 0$, then the roots are complex and conjugate to each other.

For $\Delta_2 \geq 0$, negative $A$, positive $B$, and negative $C$, the perihelion and aphelion are respectively given by

$$a\,(1-e) = r_{\rm per} = \frac{1}{u_{\rm per}} = -\tfrac{B}{2A}\left(1 - \sqrt{1 - \tfrac{4AC}{B^2}}\right), \qquad a\,(1+e) = r_{\rm ap} = \frac{1}{u_{\rm ap}} = -\tfrac{B}{2A}\left(1 + \sqrt{1 - \tfrac{4AC}{B^2}}\right), \tag{10}$$

the semimajor axis and eccentricity are given by

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[1] Computations done at 7:35:29 p.m. on 23 May 2016, Pasadena, USA, by Horizons On-Line Ephemeris System, Solar Systems Dynamic Group, Jet Propulsion Laboratory, Author: Jon.Giordini@jpl.nasa.gov.

[2] "Integration constants and mean elements for the planetary system", P. Bretagnon, *Astronomy and Astrophysics* **108**, 69-75 (1982).

[3] "Newtonian N-body calculations of the advance of Mercury's perihelion", J. V. Narlikar and N. C. Rana, *Monthly Notices of the Royal Astronomical Society* **213**, 657-663 (1985).

[4] "Precession of Mercury's Perihelion from Ranging to the MESSENGER Spacecraft", Ryan S. Park, William M. Folkner, Alexander S. Konopliv, James G. Williams, David E. Smith, and Maria T. Zuber, *The Astronomical Journal* **153**:121, 1-7 (2017).

[5] $E$ is energy, $L^2$ is squared orbital angular momentum, and $\mu \equiv Mm/(M+m)$ is reduced mass.

[6] Herbert Goldstein, *Classical Mechanics*, Second Edition, Addison-Wesley, Reading, Massachusetts 1980, pp. 85-90.

$$a \equiv \tfrac{1}{2}\left(r_{\rm ap}+r_{\rm per}\right) = -\tfrac{B}{2A}, \qquad e \equiv \frac{r_{\rm ap}-r_{\rm per}}{r_{\rm ap}+r_{\rm per}} = \sqrt{1-\tfrac{4AC}{B^2}}, \tag{11}$$

and the differential equation for the orbit, Eq. (4), becomes

$$\begin{aligned} d\theta &= -\frac{du}{\sqrt{\frac{2\mu E}{L^2}-\frac{2\mu V}{L^2}-u^2}} = -\frac{du}{\sqrt{A+Bu+Cu^2}} = -\frac{du}{\sqrt{-C}\sqrt{-\left(u^2+\frac{B}{C}u+\frac{A}{C}\right)}} = -\frac{du}{\sqrt{-C}\sqrt{\left(\frac{B}{2C}\right)^2-\frac{A}{C}-\left(u+\frac{B}{2C}\right)^2}} \\ &= -\frac{d\left[\frac{u}{\sqrt{\left(\frac{B}{2C}\right)^2-\frac{A}{C}}}\right]}{\sqrt{-C}\sqrt{1-\left[\frac{u+\frac{B}{2C}}{\sqrt{\left(\frac{B}{2C}\right)^2-\frac{A}{C}}}\right]^2}} = -\frac{d\left[\frac{u+\frac{B}{2C}}{\sqrt{\left(\frac{B}{2C}\right)^2-\frac{A}{C}}}\right]}{\sqrt{-C}\sqrt{1-\left[\frac{u+\frac{B}{2C}}{\sqrt{\left(\frac{B}{2C}\right)^2-\frac{A}{C}}}\right]^2}} = \frac{1}{\sqrt{-C}}\,d\arccos\left[\frac{u+\frac{B}{2C}}{\sqrt{\left(\frac{B}{2C}\right)^2-\frac{A}{C}}}\right]. \end{aligned} \tag{12}$$

Integration gives

$$\theta-\theta_0 = \frac{1}{\sqrt{-C}}\arccos\left[\frac{u+\frac{B}{2C}}{\sqrt{\left(\frac{B}{2C}\right)^2-\frac{A}{C}}}\right] - \frac{1}{\sqrt{-C}}\arccos\left[\frac{u_0+\frac{B}{2C}}{\sqrt{\left(\frac{B}{2C}\right)^2-\frac{A}{C}}}\right]. \tag{13}$$

We choose $\theta_0 = 0$ when $u_0 = 1/r_{\rm min}$ and the argument of the arccos in which $u_0$ appears is 1. This gives

$$\theta\sqrt{-C} = \arccos\left[\left(u+\tfrac{B}{2C}\right)/\sqrt{\left(\tfrac{B}{2C}\right)^2-\tfrac{A}{C}}\right]. \tag{14}$$

Inverse distance, $u$, is given in terms of the angle $\theta$ by[7]

$$\begin{aligned} \frac{1}{r} = u &= -\tfrac{B}{2C}+\sqrt{\left(\tfrac{B}{2C}\right)^2-\tfrac{A}{C}}\cos\left(\theta\sqrt{-C}\right) = -\tfrac{B}{2C}+\sqrt{\left(\tfrac{B}{2C}\right)^2}\sqrt{1-\tfrac{4AC}{B^2}}\cos\left(\theta\sqrt{-C}\right) \\ &= -\tfrac{B}{2C}-\tfrac{B}{2C}\sqrt{1-\tfrac{4AC}{B^2}}\cos\left(\theta\sqrt{-C}\right) = -\tfrac{B}{2C}\left[1+\sqrt{1-\tfrac{4AC}{B^2}}\cos\left(\theta\sqrt{-C}\right)\right]. \end{aligned} \tag{15}$$

Distance, $r$, is given in terms of the angle $\theta$ by

$$r = \frac{1}{u} = \frac{-\frac{2C}{B}}{1+\sqrt{1-\frac{4AC}{B^2}}\cos\left(\theta\sqrt{-C}\right)} = \frac{a\left(1-e^2\right)}{1+e\cos\left(\theta\sqrt{-C}\right)} = \frac{r_{\rm per}\left(1+e\right)}{1+e\cos\left(\theta\sqrt{-C}\right)} = \frac{r_{\rm ap}\left(1-e\right)}{1+e\cos\left(\theta\sqrt{-C}\right)}. \tag{16}$$

This equation for $r$ as a function of the angle $\theta$ is called the *orbit equation.*

Perihelia are the points closest to the Sun along the orbit, occur at minimum $r$ [the perihelion $r_{\rm per}$ in Eq. (10)], maximum $u$, $\cos\left(\theta\sqrt{-C}\right) = 1$, and $\theta\sqrt{-C} = 2\pi n$, where $n \in \{\ldots,-2,-1,0,1,2,\ldots\}$ for successive parihelia.

Aphelia are the points farthest from the Sun along the orbit, occur at maximum $r$ [the aphelion $r_{\rm ap}$ in Eq. (10)], minimum $u$, $\cos\left(\theta\sqrt{-C}\right) = -1$, and $\theta\sqrt{-C} = 2\pi n$, where $n \in \{\ldots,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\ldots\}$ for successive aphelia.

For $C = -1$, the orbit equation, Eq. (16), simplifies to

$$r = \frac{1}{u} = \frac{-\frac{2C}{B}}{1+\sqrt{1-\frac{4AC}{B^2}}\cos\theta} = \frac{a\left(1-e^2\right)}{1+e\cos\theta} = \frac{r_{\rm per}\left(1+e\right)}{1+e\cos\theta} = \frac{r_{\rm ap}\left(1-e\right)}{1+e\cos\theta} \tag{17}$$

and describes a perfectly elliptical orbit, for which the angle $\theta$ changes by exactly $2\pi$ per period $T$ between successive perihelia. For $C \neq -1$, the orbit is not a perfect ellipse and, compared to $C = -1$, successive perihelia precess at the rate

$$R_{\rm p} = \frac{2\pi}{T}\left[\frac{1}{\sqrt{-C}}-\frac{1}{\sqrt{-(-1)}}\right] = \frac{2\pi}{T}\left[\frac{1}{\sqrt{-C}}-1\right]. \tag{18}$$

For $-1 < C < 0$, $\theta$ changes by more than $2\pi$ per period $T$, the precession rate is positive, and successive perihelia advance. For $C < -1$, $\theta$ changes by less than $2\pi$ per period $T$, the precession rate is negative, and successive perihelia regress.

**For Newtonian gravity**, the central potential and force are given by

$$V(r) = -\frac{GMm}{r}, \qquad F(r) = -\frac{dV(r)}{dr} = -\frac{GMm}{r^2}, \tag{19}$$

or by

$$V\left(\tfrac{1}{u}\right) = -GMmu, \qquad F\left(\tfrac{1}{u}\right) = u^2\frac{dV\left(\frac{1}{u}\right)}{du} = -GMmu^2, \tag{20}$$

and the coefficients of Eqs. (2) through (6) are given by

$$A = 2\mu E/L^2, \qquad B = 2GMm\mu/L^2, \qquad C = -1, \qquad D = 0, \qquad \begin{array}{l}\text{Coefficients of terms beyond}\\ \text{quadratic in } u \text{ are zero}\end{array}. \tag{21}$$

[7] For the fourth equality, we take the positive square root, $\sqrt{\left(\frac{B}{2C}\right)^2} = -\frac{B}{2C}$, where we have assumed that $\frac{B}{2C}$ is negative.

Use of these coefficients in Eqs. (16), (10), and (11) gives the orbit equation

$$r=\frac{\frac{L^2}{GMm\mu}}{1+\sqrt{1+\frac{2L^2E}{G^2M^2m^2\mu}}\cos\theta}, \tag{22}$$

the perihelia and aphelia

$$a\,(1-e)\equiv r_{\rm per}=-\tfrac{GMm}{2E}\left(1-\sqrt{1+\tfrac{2L^2E}{G^2M^2m^2\mu}}\right),\qquad a\,(1+e)\equiv r_{\rm ap}=-\tfrac{GMm}{2E}\left(1+\sqrt{1+\tfrac{2L^2E}{G^2M^2m^2\mu}}\right), \tag{23}$$

and the semimajor axis and eccentricity

$$a\equiv\tfrac{1}{2}\left(r_{\rm ap}+r_{\rm per}\right)=-\tfrac{GMm}{2E},\qquad e\equiv\frac{r_{\rm ap}-r_{\rm per}}{r_{\rm ap}+r_{\rm per}}=\sqrt{1+\tfrac{2L^2E}{G^2M^2m^2\mu}}. \tag{24}$$

Since $C=-1$, **the orbit is a perfect ellipse and successive perihelia do not precess.**

The change of angle $\theta$ per period $T$ between successive perihelia is given by

$$\begin{aligned}\Delta\theta&=-\int_{u_{{\rm per}_n}}^{u_{{\rm per}_{n+1}}}\frac{du}{\sqrt{A+Bu+Cu^2+Du^3+\mathcal{E}u^4+\cdots}}\\&=\int_0^{2\pi}\frac{df\,\frac{e\sin f}{a(1-e^2)}}{\sqrt{A+B\frac{1+e\cos f}{a(1-e^2)}+C\left[\frac{1+e\cos f}{a(1-e^2)}\right]^2+D\left[\frac{1+e\cos f}{a(1-e^2)}\right]^3+\mathcal{E}\left[\frac{1+e\cos f}{a(1-e^2)}\right]^4+\cdots}},\end{aligned} \tag{25}$$

where the second equality uses the change of variable

$$u=\frac{1+e\cos f}{a(1-e^2)},\qquad du=-df\,\frac{e\sin f}{a(1-e^2)}. \tag{26}$$

This change of variable is of exactly the same form as Eq. (17) but with the angle $\theta$ replaced by an angle $f$ that, like $\theta$ in Eq. (17), changes by exactly $2\pi$ per period $T$ between successive perihelia.

For the central potentials considered below, the change $\Delta\theta$ may be expressed as a power series in the small quantity

$$\frac{G(M+m)}{c^2a}=\left(\frac{2\pi a}{cT}\right)^2\simeq 2.54989848696\times10^{-8},\quad\text{or}\quad\frac{1}{1-e^2}\frac{G(M+m)}{c^2a}=\frac{1}{1-e^2}\left(\frac{2\pi a}{cT}\right)^2\simeq 2.66247185528\times10^{-8}, \tag{27}$$

where the first equality is Kepler's third law and the second approximate equality uses $c\equiv 299792458\ \frac{\rm m}{\rm s}$ and the data of Eq. (1). For these central potentials, computation of $\Delta\theta$ up to second order in the small quantity of Eq. (27) requires retention of terms only up to quartic order in $u$. We henceforth set terms of quintic and higher order in $u$ to zero.

For $\mathcal{E}\neq0$, and with terms of quintic and higher order in $u$ set to zero, Eq. (5) may be expressed as

$$0=A+Bu+Cu^2+Du^3+\mathcal{E}u^4=\mathcal{E}(u-u_{\rm per})(u-u_{\rm ap})(u-u_3)(u-u_4)\,. \tag{28}$$

Two roots of Eq. (28) are the inverses of Mercury's empirically determined perihelion and aphelion, which are given in terms of the semimajor axis $a$ and eccentricity $e$ of Eq. (1) by

$$u_{\rm per}=\frac{1}{r_{\rm per}}=\frac{1}{a(1-e)},\qquad u_{\rm ap}=\frac{1}{r_{\rm ap}}=\frac{1}{a(1+e)}. \tag{29}$$

Comparison of coefficients on both sides of the second equality of Eq. (28) and use of Eq. (29) give

$$\begin{aligned}\frac{A}{\mathcal{E}}&=u_{\rm per}u_{\rm ap}u_3u_4=\frac{u_3u_4}{a^2(1-e^2)},\\-\frac{B}{\mathcal{E}}&=u_{\rm per}u_{\rm ap}u_3+u_{\rm per}u_{\rm ap}u_4+u_{\rm per}u_3u_4+u_{\rm ap}u_3u_4=\frac{u_3+u_4}{a^2(1-e^2)}+\frac{2u_3u_4}{a(1-e^2)},\\\frac{C}{\mathcal{E}}&=u_{\rm per}u_{\rm ap}+u_{\rm per}u_3+u_{\rm per}u_4+u_{\rm ap}u_3+u_{\rm ap}u_4+u_3u_4=\frac{1}{a^2(1-e^2)}+\frac{2\,(u_3+u_4)}{a(1-e^2)}+u_3u_4,\\-\frac{D}{\mathcal{E}}&=u_{\rm per}+u_{\rm ap}+u_3+u_4=\frac{2}{a(1-e^2)}+u_3+u_4.\end{aligned} \tag{30}$$

The last two equations of Eq. (30) give

$$u_3+u_4=-\frac{D}{\mathcal{E}}-\frac{2}{a(1-e^2)},\qquad u_3u_4=\frac{C}{\mathcal{E}}+\frac{2}{a(1-e^2)}\frac{D}{\mathcal{E}}+\frac{3+e^2}{a^2(1-e^2)^2}. \tag{31}$$

Substitution of Eq. (31) into the first two equations of Eq. (30) gives

$$Aa^2=\frac{C}{1-e^2}+\frac{2D}{a(1-e^2)^2}+\frac{(3+e^2)\mathcal{E}}{a^2(1-e^2)^3},\qquad -Ba=\frac{2C}{1-e^2}+\frac{(3+e^2)D}{a(1-e^2)^2}+\frac{(1+e^2)\mathcal{E}}{a^2(1-e^2)^3}. \tag{32}$$

The change of angle $\theta$ per period $T$ between successive perihelia is given by

$$\begin{aligned}
\Delta\theta &= -\int_{u_{\rm per_1}}^{u_{\rm per_2}} \frac{du}{\sqrt{A+Bu+Cu^2+Du^3+\mathcal{E}u^4}} = -\int_{u_{\rm per_1}}^{u_{\rm per_2}} \frac{du}{\sqrt{\mathcal{E}(u-u_{\rm per})(u-u_{\rm ap})(u-u_3)(u-u_4)}} \\
&= -\int_{u_{\rm per_1}}^{u_{\rm per_2}} \frac{du}{\sqrt{-(u_{\rm per}-u)(u-u_{\rm ap})\mathcal{E}[u_3u_4-(u_3+u_4)\,u+u^2]}} \\
&= -\int_{u_{\rm per_1}}^{u_{\rm per_2}} \frac{du}{\sqrt{\left[\frac{1}{a(1-e)}-u\right]\left[u-\frac{1}{a(1+e)}\right]\left\{-C-\frac{2D}{a(1-e^2)}-\frac{(3+e^2)\mathcal{E}}{a^2(1-e^2)^2}-\left[D+\frac{2\mathcal{E}}{a(1-e^2)}\right]u-\mathcal{E}u^2\right\}}} \\
&= \int_0^{2\pi} \frac{df\,\frac{e\sin f}{a(1-e^2)}}{\sqrt{\left[\frac{1}{a(1-e)}-\frac{1+e\cos f}{a(1-e^2)}\right]\left[\frac{1+e\cos f}{a(1-e^2)}-\frac{1}{a(1+e)}\right]\left\{-C-\frac{2D}{a(1-e^2)}-\frac{(3+e^2)\mathcal{E}}{a^2(1-e^2)^2}-\left[D+\frac{2\mathcal{E}}{a(1-e^2)}\right]\frac{1+e\cos f}{a(1-e^2)}-\mathcal{E}\left[\frac{1+e\cos f}{a(1-e^2)}\right]^2\right\}}} \\
&= \int_0^{2\pi} \frac{df}{\sqrt{1-(1+C)-\frac{3}{a(1-e^2)}D-\frac{6+e^2}{a^2(1-e^2)^2}\mathcal{E}-\left[D+\frac{4}{a(1-e^2)}\mathcal{E}\right]\frac{e\cos f}{a(1-e^2)}-\mathcal{E}\frac{e^2\cos^2 f}{a^2(1-e^2)^2}}} \\
&= \int_0^{2\pi} \frac{df}{\sqrt{1-(\Lambda+\Gamma\cos f+\Upsilon\cos^2 f)}} \\
&= \int_0^{2\pi} df\,\big[1+\tfrac{1}{2}\Lambda+\tfrac{1}{2}\tfrac{3}{4}\Lambda^2+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\Lambda^3+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}\Lambda^4+\big(\tfrac{1}{2}\Gamma+\tfrac{1}{2}\tfrac{3}{4}2\Lambda\Gamma+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}3\Lambda^2\Gamma+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}4\Lambda^3\Gamma\big)\cos f \\
&\qquad +\big(\tfrac{1}{2}\Upsilon+\tfrac{1}{2}\tfrac{3}{4}\Gamma^2+\tfrac{1}{2}\tfrac{3}{4}2\Lambda\Upsilon+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}3\Lambda^2\Upsilon+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}3\Lambda\Gamma^2+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}4\Lambda^3\Upsilon+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}6\Lambda^2\Gamma^2\big)\cos^2 f \\
&\qquad +\big(\tfrac{1}{2}\tfrac{3}{4}2\Gamma\Upsilon+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\Gamma^3+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}6\Lambda\Gamma\Upsilon+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}4\Lambda\Gamma^3+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}12\Lambda^2\Gamma\Upsilon\big)\cos^3 f \\
&\qquad +\big(\tfrac{1}{2}\tfrac{3}{4}\Upsilon^2+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}3\Lambda\Upsilon^2+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}3\Gamma^2\Upsilon+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}\Gamma^4+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}6\Lambda^2\Upsilon^2+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}12\Lambda\Gamma^2\Upsilon\big)\cos^4 f+\cdots\big] \\
&= 2\pi\big[1+\tfrac{1}{2}\Lambda+\tfrac{1}{2}\tfrac{3}{4}\Lambda^2+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\Lambda^3+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}\Lambda^4+ \\
&\qquad +\tfrac{1}{2}\big(\tfrac{1}{2}\Upsilon+\tfrac{1}{2}\tfrac{3}{4}\Gamma^2+\tfrac{1}{2}\tfrac{3}{4}2\Lambda\Upsilon+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}3\Lambda^2\Upsilon+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}3\Lambda\Gamma^2+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}4\Lambda^3\Upsilon+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}6\Lambda^2\Gamma^2\big) \\
&\qquad +\tfrac{1}{2}\tfrac{3}{4}\big(\tfrac{1}{2}\tfrac{3}{4}\Upsilon^2+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}3\Lambda\Upsilon^2+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}3\Gamma^2\Upsilon+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}\Gamma^4+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}6\Lambda^2\Upsilon^2+\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}\tfrac{7}{8}12\Lambda\Gamma^2\Upsilon\big)+\cdots\big] \\
&= 2\pi\big[1+\tfrac{1}{2}\Lambda+\tfrac{1}{2}\tfrac{3}{4}\Lambda^2+\tfrac{1}{2}\tfrac{1}{2}\Upsilon+\tfrac{1}{2}\tfrac{1}{2}\tfrac{3}{4}\Gamma^2+\cdots\big] \\
&= 2\pi\Bigg[1+\tfrac{1}{2}\Big(1+C+\tfrac{3}{a(1-e^2)}D+\tfrac{6+e^2}{a^2(1-e^2)^2}\mathcal{E}\Big)+\tfrac{1}{2}\tfrac{3}{4}\Big(1+C+\tfrac{3}{a(1-e^2)}D+\tfrac{6+e^2}{a^2(1-e^2)^2}\mathcal{E}\Big)^2 \\
&\qquad +\tfrac{1}{2}\tfrac{1}{2}\tfrac{e^2}{a^2(1-e^2)^2}\mathcal{E}+\tfrac{1}{2}\tfrac{1}{2}\tfrac{3}{4}\Big(\tfrac{e}{a(1-e^2)}D+\tfrac{4e}{a^2(1-e^2)^2}\mathcal{E}\Big)^2+\cdots\Bigg] \\
&= 2\pi\Big[1+\tfrac{1}{2}(1+C)+\tfrac{1}{2}\tfrac{3}{a(1-e^2)}D+\tfrac{1}{2}\tfrac{3}{4}(1+C)^2+\tfrac{1}{2}\tfrac{3}{4}\tfrac{6}{a(1-e^2)}(1+C)D+\tfrac{1}{2}\tfrac{1}{2}\tfrac{3}{4}\tfrac{18+e^2}{a^2(1-e^2)^2}D^2+\tfrac{1}{2}\tfrac{1}{2}\tfrac{12+3e^2}{a^2(1-e^2)^2}\mathcal{E}+\cdots\Big],
\end{aligned} \tag{33}$$

where the second equality uses Eq. (28), the fourth equality uses Eqs. (29) and (31), the fifth equality uses Eq. (26), the seventh and eleventh equalities use

$$\Lambda = 1+C+\tfrac{3}{a(1-e^2)}D+\tfrac{6+e^2}{a^2(1-e^2)^2}\mathcal{E}, \qquad \Gamma = \tfrac{e}{a(1-e^2)}D+\tfrac{4e}{a^2(1-e^2)^2}\mathcal{E}, \qquad \Upsilon = \tfrac{e^2}{a^2(1-e^2)^2}\mathcal{E}, \tag{34}$$

the eighth equality uses

$$\tfrac{1}{\sqrt{1-x}} = 1+\tfrac{1}{2}x\left(1+\tfrac{3}{4}x\left(1+\tfrac{5}{6}x\left(1+\tfrac{7}{8}x\left(1+\cdots\right)\right)\right)\right), \qquad \text{for } -1<x<1, \tag{35}$$

with $x = \Lambda+\Gamma\cos f+\Upsilon\cos^2 f$, the ninth equality uses

$$\int_0^{2\pi} df\,\cos^n f = \begin{cases} 2\pi, & \text{for } n=0, \\ 2\pi\frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{n-1}{n}, & \text{for } n\in\{2,4,6,\ldots\}, \\ 0, & \text{for } n\in\{1,3,5,\ldots\}, \end{cases} \tag{36}$$

and, for the central potentials considered below, the tenth and twelfth equalities display all terms that contribute up to second order in the small quantity of Eq. (27).

The rate of precession is given by

$$R_{\rm p} = \frac{\Delta\theta-2\pi}{T} = \frac{2\pi}{T}\Big[\tfrac{1}{2}(1+C)+\tfrac{1}{2}\tfrac{3}{a(1-e^2)}D+\tfrac{3}{8}(1+C)^2+\tfrac{3}{8}\tfrac{6}{a(1-e^2)}(1+C)D+\tfrac{3}{16}\tfrac{18+e^2}{a^2(1-e^2)^2}D^2+\tfrac{1}{4}\tfrac{12+3e^2}{a^2(1-e^2)^2}\mathcal{E}+\cdots\Big]. \tag{37}$$

For Schwarzschild's solution of general relativity,[8,9] the central potential and force are approximately given by[10]

$$V(r) = -\frac{GMm}{r} - \frac{G(M+m)L^2}{\mu c^2 r^3}, \qquad F(r) = -\frac{dV(r)}{dr} = -\frac{GMm}{r^2} - \frac{3G(M+m)L^2}{\mu c^2 r^4}, \tag{38}$$

or by

$$V\left(\tfrac{1}{u}\right) = -GMmu - \frac{G(M+m)L^2}{\mu c^2}u^3, \qquad F\left(\tfrac{1}{u}\right) = u^2\frac{dV\left(\frac{1}{u}\right)}{du} = -GMmu^2 - \frac{3G(M+m)L^2}{\mu c^2}u^4, \tag{39}$$

and the coefficients of Eqs. (2) through (5) are approximately given by

$$A = 2\mu E/L^2, \qquad B = 2GMm\mu/L^2, \qquad C = -1, \qquad D = 2G(M+m)/c^2, \qquad \begin{matrix}\text{Coefficients of terms beyond} \\ \text{cubic in } u \text{ are zero}\end{matrix}. \tag{40}$$

**The prediction of Schwarzschild's solution of general relativity for the rate of precession of Mercury's perihelion** not explained by the precession of Earth's equinoxes, motion of the ecliptic, solar oblateness, and influences of planets and asteroids on Mercury's orbit as predicted by Newton's theory of gravity **is approximately given by**

$$R_{\mathrm{p}} \simeq \frac{2\pi}{T}\left\{\frac{3}{1-e^2}\frac{G(M+m)}{c^2 a} + \frac{54+3e^2}{4(1-e^2)^2}\left[\frac{G(M+m)}{c^2 a}\right]^2\right\} = \frac{2\pi}{T}\left[\frac{3}{1-e^2}\left(\frac{2\pi a}{cT}\right)^2 + \frac{54+3e^2}{4(1-e^2)^2}\left(\frac{2\pi a}{cT}\right)^4\right] \simeq \mathbf{42.9803722063}\ldots\ \mathbf{''/century}, \tag{41}$$

where the first approximate equality uses Eq. (40) in Eq. (37), the second equality uses Eq. (27), and the third approximate equality uses $\pi \equiv (180)(60)(60)'' \equiv 648000''$, century $\equiv$ 36525 days $\equiv$ 3155760000 s, $c \equiv 299792458\ \frac{\mathrm{m}}{\mathrm{s}}$, and data of Eq. (1).

The prediction of Schwarzschild's solution of general relativity differs by roughly 3.3825 ″/century from the roughly 39.5979 ″/century of Mercury's precession not explained by the precession of Earth's equinoxes, motion of the ecliptic, solar oblateness, and influences of planets and asteroids on Mercury's orbit as predicted by Newton's theory of gravity.

An unpublished manuscript by Irving Ezra Segal[11] states that

"A conformally invariant theory of gravity which forms a natural extension of the Einstein Equivalence Principle (EEP), applies to mass points and subsumes Newtonian gravitational theory as a limiting form, is consistent both with the classical empirical tests and observational cosmology, is a component of a possible comprehensive quantum field theory and enjoys global energy conservation, is proposed."

"The theory makes the assumption of conformal invariance of the fundamental forces of nature, which is tantamount to a strengthening of the Einstein Equivalence Principle."

"This means that the observational tests of gravitational theories that follow from the Equivalence Principle (the gravitational redshift, the bending of light near massive bodies, and radar time delays) follow equally from the conformal invariance of CT. Among the classical tests of gravitational theory, this leaves only the prediction of the advance of the perihelia of the planets, the most crucial of the tests.

The gravitational potential at e.g. Mercury modifies the local physics as proposed in (20)[12] and in particular requires replacement of the nominal euclidean distance $r$ by the distance in the inertial frame of Mercury. This means that $re^f$ must be substituted for $r$, where $f$ is the gravitational potential at Mercury, or to first order in $f$, by $r\,(1+f)$.

Conventional Newtonian theory gives the following expression for the azimuthal angle $q$ of an object of mass $m$, total energy $E$, and orbital angular momentum $L$, in the central force field of a potential $V$, e.g. (36)[13]:

$$q = q_0 - \int du/\sqrt{(2mE/L^2) - (2mV/L^2) - u^2},$$

where the integration is from $u_0$ to $u$ and $u = 1/r$. With $V = -m/r$, where $m = GM$, $G$ being the Newton gravitational

[8] Hermann Weyl, *Space—Time—Matter*, Dover, New York 1952, pages 252-259.

[9] W. Pauli, *Theory of Relativity*, Dover, New York 1981, pages 164-169. Republication of a 1958 English translation.

[10] Consideration is limited to an effective one body central potential and force in which $M$ and $m$ appear symmetrically.

[11] "The Nature of Gravity", I. E. Segal (1998), previously available at http://dedekind.mit.edu/segal-archive/document.php?did=19, abstract available at http://math.mit.edu/segal-archive/scientific/nature-of-gravity.php, draft responses to referees available at http://math.mit.edu/segal-archive/scientific/replies-to-referee-re-gravity.

[12] Reference (20) of Segal is "Uber das Relativitätsprinzip und die aus demselben gezogene Folgerungen" ["On the Principle of Relativity and the Conclusions Drawn Therefrom"], A. Einstein, *Jahrbuch für Radioaktivität und Elektronik* **4**, 411-462 (1907), especially Page 457; Corrections by A. Einstein, *Jahrbuch für Radioaktivität und Elektronik* **5**, 98 (1908). For a translation, see "Einstein's comprehensive 1907 essay on relativity, Part I", H. M. Schwartz, *American Journal of Physics* **45**, 512-517 (1977); "Einstein's comprehensive 1907 essay on relativity, Part II", H. M. Schwartz, *American Journal of Physics* **45**, 811-817 (1977); "Einstein's comprehensive 1907 essay on relativity, Part III", H. M. Schwartz, *American Journal of Physics* **45**, 899-902 (1977), especially the paragraph after Eq. (30) on Page 900.

[13] Reference (36) of Segal is Lawrence G. Taff, *Celestial Mechanics: A Computational Guide for the Practitioner*, Wiley, New York 1985.

constant and $M$ the mass of the sun,[14] and the values for Mercury derived from observation via their interpretation in the Newtonian theory, e.g. (37)[15]: $a$ = semimajor axis of orbit = $57.91 \times 10^{11}$ cm; $e$ = eccentricity of orbit = 0.2056; $m = 3.334 \times 10^{26}$ gm; $m = GM = 1.325 \times 10^{26}$ cm$^3$/sec$^2$, and the substitution for $u$ of $u\,(1-f)$, where $f = m/c^2$, this expression takes the form

$$q = q_0 - \int du/\sqrt{A+Bu+Cu^2+Du^3},$$

where the constants $A$–$D$ are derived from the observed values. This differs from the prediction of GR (via the Schwarzschild solution), e.g., (8)[16], p.495, eq. (12.56), only in that $C$ is very slightly greater than $-1$. The difference of $1.6 \times 10^{-34}$ makes a contribution to the predicted perihelion advance of the same order of magnitude (in radians), on the basis of its estimation as a perturbation from the case when $D = 0$, in accordance with the classical solution[17]

$$q = q_0 - \sqrt{-C}\,\arccos\Big[-(B+2Cu)/\sqrt{B^2-4AC}\Big].$$

This is far too small to contribute measurably. The two theoretical predictions are thus equally consistent with the presently accepted emperical value.

Apart from the advance of the perihelia and applications of the Equivalence Principle, the stongest empirical support for GR is provided by the binary pulsar work of Taylor and colleagues, e.g. (38)[18] and refs. Unfortunately, this work requires a putative adaptation of GR to the two-body problem that has not been justified by rigorous mathematical estimates, e.g. (14).[19] The directly observed timing data on which this test is based have moreover been reduced on the basis of many assumptions, which derive in part from GR principles, e.g. (39)[20] and refs. A treatment of the same phenomena in CT requires non-perturbative computations together with an independent reduction of the directly observed data, and is beyond the scope of the present work."

Replacement of $r$ by $re^{\frac{G(M+m)}{c^2 r}}$ is equivalent to replacement of $\frac{1}{u}$ by $\frac{1}{u}e^{\frac{G(M+m)}{c^2}u}$, or to replacement of $u$ by

$$ue^{-\frac{G(M+m)}{c^2}u} = u\left\{1 - \frac{G(M+m)}{c^2}u + \frac{1}{2}\left[\frac{G(M+m)}{c^2}\right]^2 u^2 - \frac{1}{6}\left[\frac{G(M+m)}{c^2}\right]^3 u^3 + \cdots\right\}, \tag{42}$$

or to replacement of $u^2$ by

$$u^2 e^{-2\frac{G(M+m)}{c^2}u} = u^2\left\{1 - 2\frac{G(M+m)}{c^2}u + \frac{1}{2}\left[2\frac{G(M+m)}{c^2}\right]^2 u^2 - \frac{1}{6}\left[2\frac{G(M+m)}{c^2}\right]^3 u^3 + \cdots\right\}, \tag{43}$$

or to replacement of $du$ by

$$d\left[ue^{-\frac{G(M+m)}{c^2}u}\right] = du\left[1 - \frac{G(M+m)}{c^2}u\right]e^{-\frac{G(M+m)}{c^2}u}. \tag{44}$$

If, before replacement, the differential equation for the orbit is

$$d\theta = -\frac{du}{\sqrt{A + Bu + Cu^2}}, \tag{45}$$

then, after replacement, the new differential equation for the orbit is

$$d\theta = -\frac{du}{\sqrt{A + B_{\rm n}u + C_{\rm n}u^2 + D_{\rm n}u^3 + E_{\rm n}u^4 + \cdots}}, \tag{46}$$

and inverse distance turning points are roots of the equation

$$0 = A + B_{\rm n}u + C_{\rm n}u^2 + D_{\rm n}u^3 + E_{\rm n}u^4 + \cdots \tag{47}$$

where the new coefficients, $B_{\rm n}$, $C_{\rm n}$, $D_{\rm n}$, $E_{\rm n}$, ..., depend on in which of $du$, $u$, and $u^2$ in Eq. (45) replacements are made. We consider replacement of $u^2$ and no replacement of $du$ or $u$, as approximately required by the equivalence principle, and, for comparison, no replacements and six other combinations of replacements. For terms up to quartic in $u$, the new coefficients are given by

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[14] A potential energy $V$ does not have the same dimensions as a gravitational potential $-GM/r$. Multiplication of the sun's Newtonian gravitational potential $-GM/r$ by Mercury's mass $m$ gives the Newtonian gravitational potential energy $V = -GMm/r$.

[15] Reference (37) of Segal is C. W. Allen, *Astrophysical Quantities*, Athlone Press, London 1973.

[16] Reference (8) of Segal is C. Moller, *The Theory of Relativity*, Second Edition, Clarendon Press, Oxford 1972.

[17] This equation should be checked against Segal's manuscript (if one still exists) and references. According to Eqs. (13) and (14), $\sqrt{-C}$ should be replaced by $1/\sqrt{-C}$ and the sign of the argument of arccos should be changed.

[18] Reference (38) of Segal is "Binary pulsars and relativistic gravity", Joseph H. Taylor, Jr., *Reviews of Modern Physics* **66**, 711-719 (1994).

[19] Reference (14) of Segal is "Comments on gravitational radiation damping and energy loss in binary systems", Jürgen Ehlers, Arnold Rosenblum, Joshua N. Goldberg, and Peter Havas, *The Astrophysical Journal* **208**, L77-L81 (1976).

[20] Reference (39) of Segal is "Binary pulsars as probes of relativistic gravity", Thibault Damour, *Philosophical Transactions of the Royal Society of London A* **341**, 135-139 (1992).

| $du$ | $u$ | $u^2$ | $B_{\rm n}$ | $C_{\rm n}$ | $D_{\rm n}$ | $E_{\rm n}$ |
|---|---|---|---|---|---|---|
| No | No | No | $B$ | $C$ | $0$ | $0$ |
| **No** | **No** | **Yes** | $B$ | $C$ | $-2C\alpha$ | $2C\alpha^2$ |
| No | Yes | No | $B$ | $C-B\alpha$ | $\frac{1}{2}B\alpha^2$ | $-\frac{1}{6}B\alpha^3$ |
| No | Yes | Yes | $B$ | $C-B\alpha$ | $-2C\alpha+\frac{1}{2}B\alpha^2$ | $2C\alpha^2-\frac{1}{6}B\alpha^3$ |
| Yes | No | No | $B+4A\alpha$ | $C+4B\alpha+9A\alpha^2$ | $4C\alpha+9B\alpha^2+\frac{46}{3}A\alpha^3$ | $9C\alpha^2+\frac{46}{3}B\alpha^3+\frac{67}{3}A\alpha^4$ |
| Yes | No | Yes | $B+4A\alpha$ | $C+4B\alpha+9A\alpha^2$ | $2C\alpha+9B\alpha^2+\frac{46}{3}A\alpha^3$ | $3C\alpha^2+\frac{46}{3}B\alpha^3+\frac{67}{3}A\alpha^4$ |
| Yes | Yes | No | $B+4A\alpha$ | $C+3B\alpha+9A\alpha^2$ | $4C\alpha+\frac{11}{2}B\alpha^2+\frac{46}{3}A\alpha^3$ | $9C\alpha^2+\frac{49}{6}B\alpha^3+\frac{67}{3}A\alpha^4$ |
| Yes | Yes | Yes | $B+4A\alpha$ | $C+3B\alpha+9A\alpha^2$ | $2C\alpha+\frac{11}{2}B\alpha^2+\frac{46}{3}A\alpha^3$ | $3C\alpha^2+\frac{49}{6}B\alpha^3+\frac{67}{3}A\alpha^4$ |

(48)

where "Yes" in the first three columns designates in which of $du$, $u$, and $u^2$ replacements are made and $\alpha = G(M+m)/c^2$. If $C = -1$, as for Newtonian gravity, then the new coefficients of terms up to quartic in $u$ are given by

| $du$ | $u$ | $u^2$ | $B_{\rm n}$ | $C_{\rm n}$ | $D_{\rm n}$ | $E_{\rm n}$ |
|---|---|---|---|---|---|---|
| No | No | No | $B$ | $-1$ | $0$ | $0$ |
| **No** | **No** | **Yes** | $B$ | $-1$ | $2\alpha$ | $-2\alpha^2$ |
| No | Yes | No | $B$ | $-1-B\alpha$ | $\frac{1}{2}B\alpha^2$ | $-\frac{1}{6}B\alpha^3$ |
| No | Yes | Yes | $B$ | $-1-B\alpha$ | $2\alpha+\frac{1}{2}B\alpha^2$ | $-2\alpha^2-\frac{1}{6}B\alpha^3$ |
| Yes | No | No | $B+4A\alpha$ | $-1+4B\alpha+9A\alpha^2$ | $-4\alpha+9B\alpha^2+\frac{46}{3}A\alpha^3$ | $-9\alpha^2+\frac{46}{3}B\alpha^3+\frac{67}{3}A\alpha^4$ |
| Yes | No | Yes | $B+4A\alpha$ | $-1+4B\alpha+9A\alpha^2$ | $-2\alpha+9B\alpha^2+\frac{46}{3}A\alpha^3$ | $-3\alpha^2+\frac{46}{3}B\alpha^3+\frac{67}{3}A\alpha^4$ |
| Yes | Yes | No | $B+4A\alpha$ | $-1+3B\alpha+9A\alpha^2$ | $-4\alpha+\frac{11}{2}B\alpha^2+\frac{46}{3}A\alpha^3$ | $-9\alpha^2+\frac{49}{6}B\alpha^3+\frac{67}{3}A\alpha^4$ |
| Yes | Yes | Yes | $B+4A\alpha$ | $-1+3B\alpha+9A\alpha^2$ | $-2\alpha+\frac{11}{2}B\alpha^2+\frac{46}{3}A\alpha^3$ | $-3\alpha^2+\frac{49}{6}B\alpha^3+\frac{67}{3}A\alpha^4$ |

(49)

The new coefficients of terms of quintic and higher order in $u$ are of cubic and higher order in $\alpha = G(M+m)/c^2$.

**The prediction of the equivalence principle for the rate of precession of Mercury's perihelion** not explained by the precession of Earth's equinoxes, motion of the ecliptic, solar oblateness, and influences of planets and asteroids on Mercury's orbit as predicted by Newton's theory of gravity **is approximately given by**[21]

$$R_{\rm p} = \tfrac{\Delta\theta-2\pi}{T} = \tfrac{2\pi}{T}\left[\tfrac{1}{2}(1+C_{\rm n})+\tfrac{1}{2}\tfrac{3}{a(1-e^2)}D_{\rm n}+\tfrac{3}{8}(1+C_{\rm n})^2+\tfrac{3}{8}\tfrac{6}{a(1-e^2)}(1+C_{\rm n})D_{\rm n}+\tfrac{3}{16}\tfrac{18+e^2}{a^2(1-e^2)^2}D_{\rm n}^2+\tfrac{1}{4}\tfrac{12+3e^2}{a^2(1-e^2)^2}E_{\rm n}+\cdots\right]$$

$$\simeq \begin{cases} 0 & = \quad 0, \\ \frac{2\pi}{T}\left\{\frac{3}{1-e^2}\frac{G(M+m)}{c^2a}+\frac{30-3e^2}{4(1-e^2)^2}\left[\frac{G(M+m)}{c^2a}\right]^2\right\} = \frac{2\pi}{T}\left[\frac{3}{1-e^2}\left(\frac{2\pi a}{cT}\right)^2+\frac{30-3e^2}{4(1-e^2)^2}\left(\frac{2\pi a}{cT}\right)^4\right] & \simeq \mathbf{42.9803698934\ldots''/century}, \\ -\frac{2\pi}{T}\left\{\frac{1}{1-e^2}\frac{G(M+m)}{c^2a}-\frac{1}{(1-e^2)^2}\left[\frac{G(M+m)}{c^2a}\right]^2\right\} = -\frac{2\pi}{T}\left[\frac{1}{1-e^2}\left(\frac{2\pi a}{cT}\right)^2-\frac{1}{(1-e^2)^2}\left(\frac{2\pi a}{cT}\right)^4\right] & \simeq -14.3267886334\ldots''/\text{century}, \\ \frac{2\pi}{T}\left\{\frac{2}{1-e^2}\frac{G(M+m)}{c^2a}+\frac{10+e^2}{4(1-e^2)^2}\left[\frac{G(M+m)}{c^2a}\right]^2\right\} = \frac{2\pi}{T}\left[\frac{2}{1-e^2}\left(\frac{2\pi a}{cT}\right)^2+\frac{10+e^2}{4(1-e^2)^2}\left(\frac{2\pi a}{cT}\right)^4\right] & \simeq 28.6535789874\ldots''/\text{century}, \\ -\frac{2\pi}{T}\left\{\frac{2}{1-e^2}\frac{G(M+m)}{c^2a}-\frac{6-15e^2}{4(1-e^2)^2}\left[\frac{G(M+m)}{c^2a}\right]^2\right\} = -\frac{2\pi}{T}\left[\frac{2}{1-e^2}\left(\frac{2\pi a}{cT}\right)^2-\frac{6-15e^2}{4(1-e^2)^2}\left(\frac{2\pi a}{cT}\right)^4\right] & \simeq -28.6535775180\ldots''/\text{century}, \\ \frac{2\pi}{T}\left\{\frac{1}{1-e^2}\frac{G(M+m)}{c^2a}+\frac{3+31e^2}{(1-e^2)^2}\left[\frac{G(M+m)}{c^2a}\right]^2\right\} = \frac{2\pi}{T}\left[\frac{1}{1-e^2}\left(\frac{2\pi a}{cT}\right)^2+\frac{3+31e^2}{(1-e^2)^2}\left(\frac{2\pi a}{cT}\right)^4\right] & \simeq 14.3267906592\ldots''/\text{century}, \\ -\frac{2\pi}{T}\left\{\frac{3}{1-e^2}\frac{G(M+m)}{c^2a}-\frac{126+3e^2}{4(1-e^2)^2}\left[\frac{G(M+m)}{c^2a}\right]^2\right\} = -\frac{2\pi}{T}\left[\frac{3}{1-e^2}\left(\frac{2\pi a}{cT}\right)^2-\frac{126+3e^2}{4(1-e^2)^2}\left(\frac{2\pi a}{cT}\right)^4\right] & \simeq -42.9803550170\ldots''/\text{century}, \\ 0 & = \quad 0, \end{cases} \tag{50}$$

where the second equality replaces $B$, $C$, $D$, $\mathcal{E}$ by $B_{\rm n}$, $C_{\rm n}$, $D_{\rm n}$, $E_{\rm n}$ in Eq. (37), the third approximate equality uses $B_{\rm n}$, $C_{\rm n}$, $D_{\rm n}$, $E_{\rm n}$ from the respective case of Eq. (49), uses Eq. (32) to eliminate $A$ and $B$ as necessary, and uses the expansion

$$\tfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots, \tag{51}$$

the fourth equality uses Eq. (27), and the fifth approximate equality uses $\pi \equiv (180)(60)(60)'' \equiv 648000''$, century $\equiv$ 36525 days $\equiv$ 3155760000 s, $c \equiv 299792458\ \frac{\rm m}{\rm s}$, and the data of Eq. (1).

The rate of precession of Eq. (50) is less than the rate of precession of Eq. (41) by only 0.0000023128... ″/century. This difference is much smaller than differences of the predictions from 39.5979 ″/century and is much smaller than the empirical uncertainty. **The approximate prediction of the equivalence principle is as consistent as the approximate prediction of Schwarzschild's solution of general relativity with the rate of precession of Mercury's perihelion** not explained by the precession of Earth's equinoxes, motion of the ecliptic, solar oblateness, and influences of planets and asteroids on Mercury's orbit as predicted by Newton's theory of gravity.

[21] Replacement of $du$, $u$, and $u^2$, that is, a name change of the integration variable, cannot change the result. As a check of the first line of Eq. (50), the ninth line has been verified up to second order in the small quantity of Eq. (27).