Irving Ezra Segal's estimate of the radius of the Universe

According to Segal's chronometric cosmology (CC),[1,2] the relation of redshift, z, and proper motion, μ, to the radius of the Universe, R, speed of separation at the source (divided by c), β, and direction of motion, θ, is

$$\frac{R}{c}2\mu\sqrt{z} = \frac{R}{c}v = \frac{\beta\sin\theta}{1-\beta\cos\theta}. \tag{1}$$

Table 1 shows observed redshift, z, and observed proper motion, μ,[3] for 24 sources[1] on which Segal bases estimates of R, β, and, for individual sources, θ. The 24 sources are ordered by values of $2\mu\sqrt{z}$. Segal reports the estimates

$$R = 160 \pm 40 \text{ Mpc}, \qquad \beta = 0.957. \tag{2}$$

The fit below, to the same data, gives the estimates

$$R \simeq 504.564\ldots \text{ million lightyears} = 154.700\ldots \text{ million parsecs}, \qquad \beta \simeq 0.9577\ldots. \tag{3}$$

The difference of 160 Mpc from 154.7 Mpc is well within Segal's estimated uncertainty of (at most) 40 Mpc.

Table 1

Source	z	μ in $\frac{\text{mas}}{\text{yr}}$	$2\mu\sqrt{z}$ in $\frac{\text{mas}}{\text{yr}}$	$\frac{R}{c}2\mu\sqrt{z}$ for $R = 504.564\ldots$ million lightyears	Cumulative frequency
3C84	0.0172	0.24	0.062951...	0.153991...	1/48
1951+498	0.466	0.07	0.095569...	0.233782...	3/48
3C263	0.652	0.06	0.096895...	0.237026...	5/48
3C245	1.029	0.11	0.223167...	0.545911...	7/48
0735+178	0.424	0.18	0.234415...	0.573425...	9/48
4C39.25	0.699	0.16	0.267539...	0.654455...	11/48
0850+581	1.322	0.12	0.275947...	0.675023...	13/48
0212+735	2.367	0.09	0.276931...	0.677428...	15/48
3C216	0.669	0.17	0.278094...	0.680273...	17/48
1150+812	1.25	0.13	0.290688...	0.711082...	19/48
OJ287	0.306	0.28	0.309776...	0.757775...	21/48
4C34.47	0.206	0.36	0.326788...	0.799388...	23/48
NRA0140	1.258	0.15	0.336481...	0.823101...	25/48
3C179	0.846	0.19	0.349517...	0.854989...	27/48
BLLac	0.0695	0.76	0.400715...	0.980229...	29/48
1642+690	0.751	0.34	0.589289...	1.441520...	31/48
2251+158	0.859	0.35	0.648775...	1.587034...	33/48
1928+738	0.302	0.6	0.659454...	1.613156...	35/48
3C279	0.538	0.5	0.733484...	1.794249...	37/48
3C345	0.595	0.48	0.740507...	1.811429...	39/48
3C273	0.158	1.20	0.953981...	2.333627...	41/48
3C120	0.033	2.66	0.966425...	2.364070...	43/48
3C395	0.635	0.64	1.019992...	2.495103...	45/48
CTA102	1.037	0.65	1.323831...	3.238355...	47/48

If R, like c, is assumed to be constant and if the probability density distribution for velocity of separation at the source (divided by c) $\boldsymbol{\beta}$ is isotropic (does not depend on direction), then the n-th moment of Eq. (1) is

$$\frac{R^n}{c^n}\left\langle\left(2\mu\sqrt{z}\right)^n\right\rangle = \left\langle\left(\frac{\beta\sin\theta}{1-\beta\cos\theta}\right)^n\right\rangle = \int_0^1 d\beta\ \phi(\beta)\int_0^\pi d\theta\ \tfrac{1}{2}\sin\theta\left(\frac{\beta\sin\theta}{1-\beta\cos\theta}\right)^n, \tag{4}$$

where $\phi(\beta)$ is a weight function that may depend on the magnitude β of $\boldsymbol{\beta}$. With integrals over θ evaluated in the appendix, the first and second moments are

$$\frac{R}{c}\left\langle 2\mu\sqrt{z}\right\rangle = \left\langle\frac{\beta\sin\theta}{1-\beta\cos\theta}\right\rangle = \int_0^1 d\beta\ \phi(\beta)\int_0^\pi d\theta\ \tfrac{1}{2}\sin\theta\frac{\beta\sin\theta}{1-\beta\cos\theta} = \int_0^1 d\beta\ \phi(\beta)\frac{\pi}{2\beta}\left(1-\sqrt{1-\beta^2}\right), \tag{5}$$

1 "Apparent superluminal sources, comparative cosmology and the cosmic distance scale", I. E. Segal, *Monthly Notices of the Royal Astronomical Society* **242**, 423-427 (1990), http://articles.adsabs.harvard.edu/cgi-bin/nph-iarticle_query?1990MNRAS.242..423S&defaultprint=YES&filetype=.pdf.

2 "APPARENT SUPERRELATIVISTIC VELOCITIES AND THE CHRONOMETRIC COSMOLOGY", I. E. Segal, *The Astrophysical Journal* **227**, 15-17 (1979).

3 $1 \equiv \text{radian} \equiv \frac{180}{\pi}{}^\circ \equiv \frac{180(60)}{\pi}{}' \equiv \frac{180(60)^2}{\pi}{}'' \equiv \frac{180(60)^2 1000}{\pi}$ mas $\simeq 206264806\ldots$ milliarcseconds.

$$\frac{R^2}{c^2}\left\langle\left(2\mu\sqrt{z}\right)^2\right\rangle=\left\langle\left(\frac{\beta\sin\theta}{1-\beta\cos\theta}\right)^2\right\rangle=\int_0^1\! d\beta\;\phi(\beta)\int_0^\pi\! d\theta\;\tfrac{1}{2}\sin\theta\left(\frac{\beta\sin\theta}{1-\beta\cos\theta}\right)^2=\int_0^1\! d\beta\;\phi(\beta)\left[\frac{1}{\beta}\ln\!\left(\frac{1+\beta}{1-\beta}\right)-2\right]. \tag{6}$$

If speed of separation at the source β is assumed to be constant, $\phi(\hat{\beta})=\delta(\hat{\beta}-\beta)$, then Eqs. (4), (5), and (6) reduce to

$$\frac{R^n}{c^n}\left\langle\left(2\mu\sqrt{z}\right)^n\right\rangle=\int_0^\pi\! d\theta\;\tfrac{1}{2}\sin\theta\left(\frac{\beta\sin\theta}{1-\beta\cos\theta}\right)^n, \tag{7}$$

$$\frac{R}{c}\left\langle 2\mu\sqrt{z}\right\rangle=\frac{\pi}{2\beta}\left(1-\sqrt{1-\beta^2}\right), \tag{8}$$

$$\frac{R^2}{c^2}\left\langle\left(2\mu\sqrt{z}\right)^2\right\rangle=\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-2. \tag{9}$$

Equations (8) and (9) allow estimates of R and β to be obtained from the observed averages $\langle 2\mu\sqrt{z}\rangle$ and $\langle(2\mu\sqrt{z})^2\rangle$. For the sample of observed values of $2\mu\sqrt{z}$ in Table 1, the observed averages $\langle 2\mu\sqrt{z}\rangle$ and $\langle(2\mu\sqrt{z})^2\rangle$ are

$$\left\langle 2\mu\sqrt{z}\right\rangle=0.477550\ldots\ \tfrac{\text{mas}}{\text{yr}}, \tag{10}$$

$$\left\langle\left(2\mu\sqrt{z}\right)^2\right\rangle=0.334940\ldots\ \tfrac{\text{mas}^2}{\text{yr}^2}. \tag{11}$$

Elimination of R from Eqs. (8) and (9) and use of Eqs. (10) and (11) gives

$$\sqrt{\frac{1}{\beta}\ln\!\left(\frac{1+\beta}{1-\beta}\right)-2}\left[\frac{\pi}{2\beta}\left(1-\sqrt{1-\beta^2}\right)\right]^{-1}=\sqrt{\left\langle\left(2\mu\sqrt{z}\right)^2\right\rangle}\left\langle 2\mu\sqrt{z}\right\rangle^{-1}=\frac{\sqrt{0.334940\ldots\ \frac{\text{mas}^2}{\text{yr}^2}}}{0.477550\ldots\ \frac{\text{mas}}{\text{yr}}}=1.211892\ldots. \tag{12}$$

Comparison with the table

β	$\dfrac{\sqrt{\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-2}}{\frac{\pi}{2\beta}\left(1-\sqrt{1-\beta^2}\right)}$
0.9570	1.210490...
0.9576	1.211655...
0.9577	1.211851...
0.9578	1.212048...
0.9580	1.212443...

gives the estimate

$$\beta\simeq 0.9577\ldots. \tag{13}$$

This estimate is also obtained by comparison of the standard deviation of $2\mu\sqrt{z}$ divided by $\langle 2\mu\sqrt{z}\rangle$,

$$\sqrt{\frac{\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-2}{\left[\frac{\pi}{2\beta}\left(1-\sqrt{1-\beta^2}\right)\right]^2}-1}=\frac{\sigma}{\langle 2\mu\sqrt{z}\rangle}=\sqrt{\frac{\left\langle\left(2\mu\sqrt{z}\right)^2\right\rangle}{\left\langle 2\mu\sqrt{z}\right\rangle^2}-1}=0.684605\ldots, \tag{12$'$}$$

with the table

β	$\sqrt{\dfrac{\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-2}{\left[\frac{\pi}{2\beta}\left(1-\sqrt{1-\beta^2}\right)\right]^2}-1}$
0.9570	0.682119...
0.9576	0.684185...
0.9577	0.684532...
0.9578	0.684880...
0.9580	0.685578...

Comparison of Eq. (12′) is about three times more sensitive than comparison of Eq. (12) to changes in β in this range. Segal compares Eq. (12′) but rounds to the estimate $\beta=0.957$ and away from the closer estimate $\beta=0.958$. Segal also estimates that the "overall effective uncertainty in β on the basis of the CC and the sample of Cohen *et al.* appears to be comparable to a standard deviation of the order of 2 per cent."

Use of Eqs. (10) and $\beta=0.9577$ in Eq. (8) gives the following rough estimate for the radius of the Universe:

$$\begin{aligned}R&\simeq\frac{c}{\langle 2\mu\sqrt{z}\rangle}\frac{\pi}{2\beta}\left(1-\sqrt{1-\beta^2}\right)\simeq\frac{c}{0.477550\ldots\ \frac{\text{mas}}{\text{yr}}}\frac{\pi}{2\,(0.9577)}\left(1-\sqrt{1-(0.9577)^2}\right)\\&=\frac{c}{0.477550\ldots\ \frac{\text{mas}}{\text{yr}}}\frac{\pi}{2\,(0.9577)}\left(1-\sqrt{1-(0.9577)^2}\right)(206264806\ldots\ \text{mas})\\&=504.564\ldots\ \text{million lightyears}=154.700\ldots\ \text{million parsecs.}\end{aligned} \tag{14}$$

I do not know how Segal obtained the higher estimate $R = 160$ Mpc. Segal states that "A first-order perturbative stochastic analysis indicates that the standard error of this estimate is at most 40 Mpc." The difference of $R = 160$ Mpc from $R = 154.7$ Mpc is well within 40 Mpc.

Segal mentions that the hypothesis that the values of β are substantially uniform is "the simplest *a priori* hypothesis", "is moreover suggested by the appearance of a common physical mechanism underlying most superluminal motions, and is in any event subject to an *a posteriori* statistical test." The *a posteriori* statistical test compares the cumulative frequency distribution of $\frac{R}{c}2\mu\sqrt{z}$ for the observed z and μ and constant R from Eq. (14) with the cumulative probability distribution of $\frac{\beta\sin\theta}{1-\beta\cos\theta}$ predicted by isotropy of $\boldsymbol{\beta}$ and constant β from Eq. (13).

For $R = 504.564\ldots$ million lightyears, the observed values of $2\mu\sqrt{z}$ in Table 1 are converted to dimensionless values of $\frac{R}{c}2\mu\sqrt{z}$ according to

$$\frac{R}{c}2\mu\sqrt{z} = \frac{504.564\ldots \text{ million lightyears}}{206264806\ldots \text{ mas}}\frac{1}{c}2\mu\sqrt{z} = \left(2.446199\ldots \frac{\text{yr}}{\text{mas}}\right)2\mu\sqrt{z}. \tag{15}$$

Table 1 shows these values along with their observed cumulative frequencies. The cumulative frequencies are the central points of 24 bins of width 1/24 from 0 to 1, one for each source. This cumulative frequency distribution is plotted as dots in Figure 1.

For constant β, the maximum value of $\frac{\beta\sin\theta}{1-\beta\cos\theta}$ is $\beta/\sqrt{1-\beta^2}$, which occurs when $\cos\theta = \beta$. The value of $\frac{\beta\sin\theta}{1-\beta\cos\theta}$ declines as θ varies in either direction away from $\theta = \arccos\beta$. The cumulative probability that $\frac{\beta\sin\theta}{1-\beta\cos\theta}$ is less than some value x is

$$\text{Prob}[\beta\sin\theta/(1-\beta\cos\theta) < x] = 1 - \int_{\theta_1}^{\theta_2} d\theta\,\tfrac{1}{2}\sin\theta = 1 + \tfrac{1}{2}(\cos\theta_2 - \cos\theta_1) = 1 - \sqrt{\beta^2(1+x^2) - x^2}\big/\left[\beta\left(1+x^2\right)\right], \tag{16}$$

where θ_1 and θ_2 are solutions of $\frac{\beta\sin\theta_j}{1-\beta\cos\theta_j} = x$. For $\beta = 0.9577$, this cumulative probability distribution is plotted as a solid curve in Figure 1 and gives probability 1 when $\frac{\beta\sin\theta}{1-\beta\cos\theta}$ takes its maximum value

$$\beta/\sqrt{1-\beta^2} = 0.9577/\sqrt{1-(0.9577)^2} \simeq 3.328022\ldots. \tag{17}$$

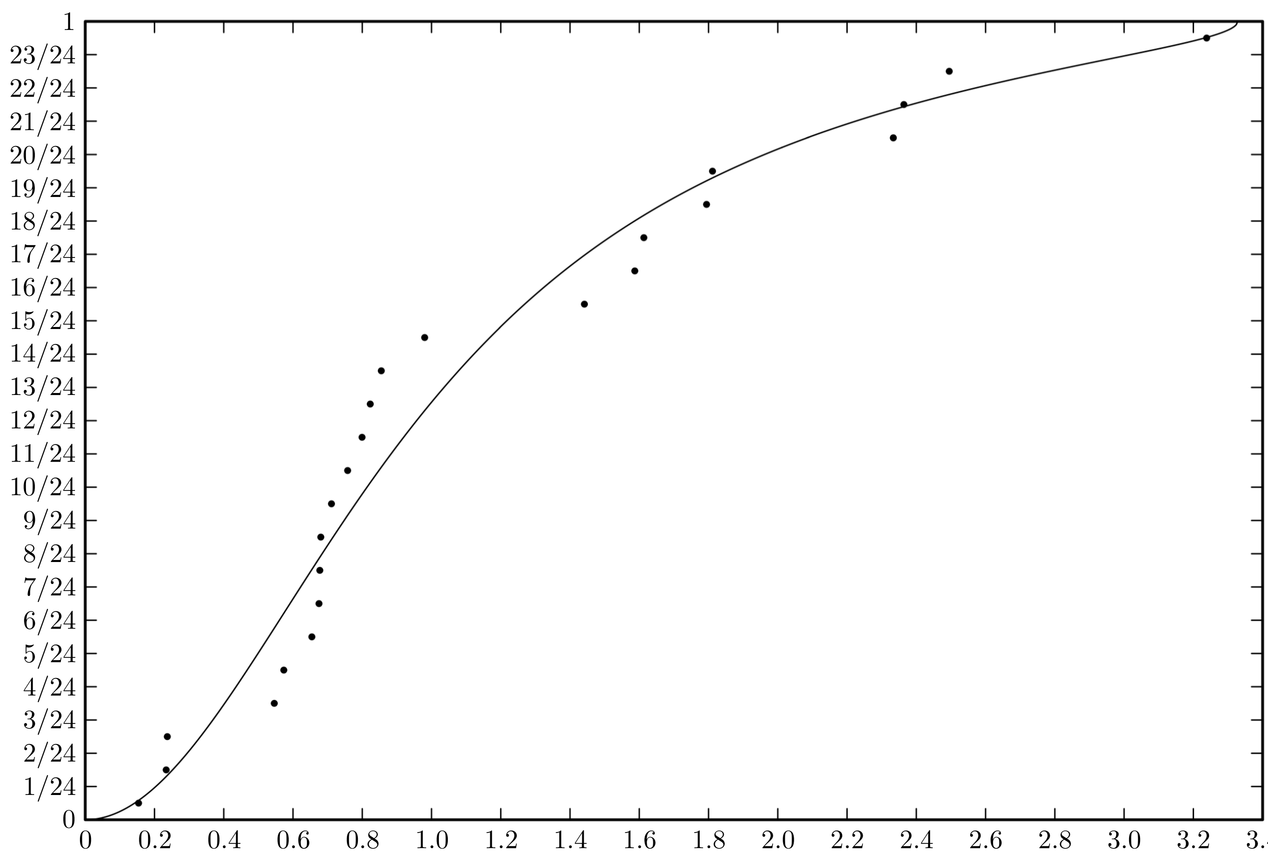

Figure 1. Observed cumulative frequency versus $\frac{R}{c}2\mu\sqrt{z}$ for $R \simeq 504.564\ldots$ million lightyears from Table 1 (dots) and theoretical cumulative probability of Eq. (16) versus x for $\beta = 0.9577$ (solid curve).

Although Segal uses slightly different values of R and β, his comment, "The fit appears quite acceptable", still applies. [Note that the value $\beta \simeq 0.9577\ldots$ is from Eq. (13), not from a separate fit of Eq. (16) to the dots.] Segal also states "that allowance for the fitting of two parameters from the data would be unlikely to alter the apparent acceptability." Deviations might be due to ordinary statistical variation of the directions of motion, θ, and to speeds of separation at the source, β, that are only approximately constant.

Segal states that "For a test that goes beyond the present phenomenological study, as well as for more precise estimates of the important astrophysical parameters estimated here, a larger sample that is objectively delineated and systematically observed would be extremely valuable." Such a larger sample of observed right ascension, declination, redshift, and proper motion data might allow estimation of parameters in models that describe deviations of the shape of the spatial Universe from a 3-sphere of constant radius R and volume $2\pi^2 R^3$. Can such a larger sample be obtained from existing surveys?

Segal's article (referenced in Footnote 1) contains several errors, which might have caused some reluctance to accept its rough estimates and suggestions for further study. On Page 423, $v = v(\beta,\theta) = \beta \sin\theta\,(1-\beta\cos\theta)$ should read $v = v(\beta,\theta) = \beta\sin\theta/(1-\beta\cos\theta)$. On Page 423, $\langle v\rangle = (1/2\pi)^{-1}\left[1-\left(1-\beta^2\right)^{1/2}\right]$ should read $\langle v\rangle = \pi/(2\beta)\left[1-\left(1-\beta^2\right)^{1/2}\right]$. On Page 424, the column of Table 1 labeled $v_{\max}$ lists rounded values of $\beta/\sqrt{1-\beta}$, not rounded values of $\beta/\sqrt{1-\beta^2}$. On Page 425, the factor 63.27 should read 63.24. On Page 425, within the integrand in Eq. (2) and in a later equation on Page 425, the factor of $\cos\theta$ should read $\sin\theta$. On Page 426, in Table 2, for source 3C395, the value of v should be more than 2.5, not 1.60.

Appendix. Two integrals over θ

For the integral

$$\int_0^\pi d\theta\, \tfrac{1}{2}\sin\theta\left(\frac{\beta\sin\theta}{1-\beta\cos\theta}\right)^2 = -\frac{1}{2\beta}\int_{1-\beta}^{1+\beta} du\left(\frac{1-\beta^2}{u^2}-\frac{2}{u}+1\right) = \frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-2 \tag{A1}$$

the first equality uses $u = 1-\beta\cos\theta$.

An integral table provides the integral[4]

$$\int_0^\pi \frac{\sin^2 x}{p+q\cos x}dx = \frac{p\pi}{q^2}\left(1-\sqrt{1-\frac{q^2}{p^2}}\right). \tag{A2}$$

The substitutions $x=\theta$, $p=1$, and $q=-\beta$ and multiplication by $\frac{\beta}{2}$ give the same result that I derived on 22 February 2014,

$$\begin{aligned}
\int_0^\pi d\theta\, \tfrac{1}{2}\sin\theta\frac{\beta\sin\theta}{1-\beta\cos\theta} &= \frac{1}{4}\int_{-\pi}^{\pi} d\theta\,\frac{\beta\sin^2\theta}{1-\beta\cos\theta} = -\frac{1}{4}\oint\frac{dz}{iz}\,\frac{\frac{\beta}{4}\left(z-\frac{1}{z}\right)^2}{1-\frac{\beta}{2}\left(z+\frac{1}{z}\right)} = \frac{1}{8i}\oint dz\,\frac{\left(z^2-1\right)^2}{z^2\left(z^2-\frac{2}{\beta}z+1\right)}\\
&= \frac{1}{8i}\oint dz\,\frac{\left(z^2-1\right)^2}{z^2\left[z-\frac{1}{\beta}\left(1-\sqrt{1-\beta^2}\right)\right]\left[z-\frac{1}{\beta}\left(1+\sqrt{1-\beta^2}\right)\right]}\\
&= \frac{2\pi i}{8i}\left\{\lim_{z\to0}\frac{d}{dz}\frac{\left(z^2-1\right)^2}{z^2-\frac{2}{\beta}z+1} + \lim_{z\to\frac{1}{\beta}\left(1-\sqrt{1-\beta^2}\right)}\frac{\left(z^2-1\right)^2}{z^2\left[z-\frac{1}{\beta}\left(1+\sqrt{1-\beta^2}\right)\right]}\right\}\\
&= \frac{\pi}{8\beta\sqrt{1-\beta^2}}\,\frac{4\sqrt{1-\beta^2}\left(1-\sqrt{1-\beta^2}\right)^2-\left[\left(1-\sqrt{1-\beta^2}\right)^2-\beta^2\right]^2}{\left(1-\sqrt{1-\beta^2}\right)^2}\,\frac{\left(1+\sqrt{1-\beta^2}\right)^2}{\left(1+\sqrt{1-\beta^2}\right)^2}\\
&= \frac{\pi}{2\beta}\left(1-\sqrt{1-\beta^2}\right), \qquad \text{for } 0\le\beta<1.
\end{aligned} \tag{A3}$$

The first equality uses the fact that the integrand is an even function of θ. The second equality uses $z = e^{i\theta} = \cos\theta + i\sin\theta$. The fifth equality uses the residue theorem and the fact that, for $0<\beta<1$, the integrand is analytic within the unit circle except for a second order pole at $z=0$ and a first order pole at $z=\frac{1}{\beta}\left(1-\sqrt{1-\beta^2}\right)$. Algebra of the later equalities simplifies by use of $\left(1-\sqrt{1-\beta^2}\right)\left(1+\sqrt{1-\beta^2}\right)=\beta^2$.

[4] This is Equation 3.644-4 on Page 402 of I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Seventh Edition, Alan Jeffrey and Daniel Zwillinger, Editors, Elsevier Academic Press, Burlington, Massachusetts 2007.